Conformal dimension of subcritical and critical exponential metrics

Abstract

We prove that the ordinary conformal dimension of the subcritical and critical exponential metrics on $\mathbb{R}^d$ is d, for every $d \geq 2$ and $0 < \gamma \leq \sqrt{2d}$. We use the existing metric constructions and critical-topology theorem, with precise references to the higher-dimensional results. A stationary diffusion treats the subcritical and critical cases in the same way. Its exact exponential decay rate supplies the strict inequality needed to construct, for every $p > d$, one globally quasisymmetric metric with zero p-dimensional Hausdorff measure. We include the finite-path proof for the weighted graph, the extension to all weak subcritical metrics, and the transfer to the γ-LQG sphere, $0 < \gamma < 2$, whose conformal dimension is 2. The metric, Gaussian multiplicative chaos, and Gaussian maximum results used as foundations are stated explicitly.

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1 Statement and scope

A homeomorphism $f : (X, D) \to (Y, \delta)$ is *quasisymmetric* if an increasing homeomorphism $\eta : [0, \infty) \to [0, \infty)$ satisfies

$$\frac{\delta(f(x), f(y))}{\delta(f(x), f(z))} \le \eta\left(\frac{D(x, y)}{D(x, z)}\right) \qquad (x, y, z \text{ distinct}) .$$

The ordinary conformal dimension is

$$\mathrm{Cdim}(X, D) = \inf\{\dim_{\mathrm{H}}(X, \delta) : \mathrm{id} : (X, D) \to (X, \delta) \text{ is quasisymmetric}\} . \tag{1.1}$$

There is no Ahlfors regularity requirement in this definition.

Fix $d \ge 2$, and let h be the whole-space log-correlated Gaussian field, modulo constants, with covariance $-\log|x - y|$. In dimension two this is the whole-plane Gaussian free field; in higher dimensions it is a different field. Set

$$0 < \gamma \le \gamma_{\mathrm{c}} := \sqrt{2d}, \qquad Q = \frac{d}{\gamma} + \frac{\gamma}{2}, \qquad a = Q - \gamma \ge 0, \qquad s = \frac{\gamma}{\xi}, \tag{1.2}$$

Here $\xi > 0$ is the Weyl parameter of the corresponding exponential metric, so $Q_d(\xi) = Q$. At $\gamma = \gamma_{\mathrm{c}}$ we take the critical parameter

$$\xi_{\mathrm{c}} = \sup\{t > 0 : Q_d(t) > \sqrt{2d}\} .$$

The critical-topology theorem applies at this endpoint. This convention also specifies the metric if Q_d has a plateau at $\sqrt{2d}$. The number s is initially a scaling exponent. No identification of s with the critical Hausdorff dimension is assumed.

For a length metric D, write $D(x, y; U)$ for its internal metric in an open set U. If f is continuous, $e^{\xi f} \cdot D$ denotes the length metric obtained by integrating $e^{\xi f}$ along D-rectifiable paths. Write B_r^E and B_r^D for Euclidean and metric balls.

Proposition 1.1 (Metric inputs)**.** *The subcritical and critical metric assignment $h \mapsto D_h$ is measurable and local, and is defined for a log-correlated field plus a continuous function. It has the following properties.*

(i) *D_h is a finite length metric inducing the Euclidean topology.*

(ii) *Continuous Weyl scaling holds simultaneously on the continuous-perturbation orbit. Translation covariance holds for each fixed translation. The dilation and additive-constant rules hold simultaneously on each full dilation and additive-constant orbit:*

$$D_{h+f} = e^{\xi f} \cdot D_h, \qquad D_h(rx, ry) = D_{h(r\cdot)+Q\log r}(x, y) \quad (r > 0) . \tag{1.3}$$

(iii) *Distances across fixed annuli have every negative moment after normalization by $r^{\xi Q} e^{\xi h_r(z)}$. Local crossing and connection costs have every positive moment with the same normalization. A connection cost joins any two crossings of a smaller fixed annulus inside a fixed enlarged annulus. All constants are uniform in the center z and scale r. Here $h_r(z)$ may be a spherical average or any fixed Gaussian local average whose difference from it has uniformly bounded variance.*

The same statements apply to internal metrics and to the local versions of the field used below.

Proof. In dimension two, the construction and covariance are supplied by [8, 4], and critical finiteness and Euclidean topology by [3]. In dimensions $d \geq 3$, the subcritical strong assignment is constructed in [9, Definition 1.1 and Theorem 1.2]. The positive-Q construction [10, Definition 1.1 and Theorem 1.2] supplies locality, the length property, simultaneous continuous Weyl scaling, exact fixed similarities, and uniqueness throughout $Q > 0$. At $\xi = \xi_c$, finiteness at every point and Euclidean topology follow from [11, Theorem A]. Uniqueness identifies the metric in that theorem with the positive-Q assignment, up to deterministic normalization.

Subcritical crossing and connection moments follow from [2, Propositions 4.7 and 4.10]. In the planar critical case, both tails follow from [3, Lemma 2.1 and the paragraph following it]; a separating loop connects any two annular crossings. In dimensions $d \geq 3$, every negative crossing moment is given by [10, Lemma 13.2]. At criticality, [11, Proposition 3.2] gives a local crossing and a separator whose normalized length and intrinsic diameter have tail at most $C \exp(-cu^2/\log u)$ at height e^u. This implies every positive moment. The separator's tile-boundary routes stay in doubled tile buffers near its annulus, so their connecting paths lie in a fixed enlarged annulus. Every crossing meets the separator, and its intrinsic diameter bounds the connection cost. Trimming a crossing at its last inner-boundary visit gives the same annular locality for the crossing cost. Exact scaling and Gaussian coarse oscillations give the stated uniformity and the permitted averaging conventions.

Finally, Proposition 10.2, applied directly with $C = 1$, supplies the simultaneous dilation and additive-constant version from covariance at each fixed dilation, without changing D_h. □

The connection estimate is the buffered form of an *around* estimate. It does not require every positive moment of a Euclidean ball's metric diameter. Section 10 also proves a bi-Lipschitz reduction for weak subcritical metrics.

Theorem 1.2. *For every $d \geq 2$ and $0 < \gamma \leq \sqrt{2d}$, let D_h be the corresponding subcritical or critical exponential metric, with the parameter convention above. Then almost surely*

$$\operatorname{Cdim}(\mathbb{R}^d, D_h) = d\,. \tag{1.4}$$

More precisely, for each fixed $p > d$ there is almost surely a metric δ_p on all of $\mathbb{R}^d$ such that the identity from D_h to δ_p is globally quasisymmetric and $\mathcal{H}^p_{\delta_p}(\mathbb{R}^d) = 0$.

The probabilistic assertion is for each fixed choice of parameters. The theorem does not assert a common exceptional event for every γ, attainment of the infimum in (1.1), or quasisymmetry to Euclidean distance.

Corollary 1.3 (All subcritical dimensions)**.** *For every $d \geq 2$ and every weak subcritical γ-exponential metric, $0 < \gamma < \sqrt{2d}$, almost surely* $\operatorname{Cdim}(\mathbb{R}^d, D_h) = d$. *The weak-metric properties and their reduction to those of Proposition 1.1 are given in Section 10.*

Corollary 1.4 (Planar metrics and spheres)**.** *For the canonical planar γ-LQG metric, $0 < \gamma \leq 2$, almost surely* $\operatorname{Cdim}(\mathbb{R}^2, D_h) = 2$. *For the intrinsic γ-LQG sphere of any fixed positive area, $0 < \gamma < 2$, almost surely its ordinary conformal dimension is 2.*

The sphere statement uses the standard three-insertion representation from [1]. A critical $\gamma = 2$ sphere is not included in Corollary 1.4.

Outline of the proof. The argument follows the annular-weight construction used by Miller and Tian for the Brownian sphere [12]. We prove the geometric and graph arguments needed here. The new probabilistic calculation is governed by

$$\lambda_a(p) = \sqrt{a^2 + 2p} - a, \qquad \lambda_a(d) = \gamma, \qquad \lambda_a(p)/\xi > s \quad (p > d)\,. \tag{1.5}$$

We use the same stationary diffusion for $a > 0$ and $a = 0$. Estimates at a root sampled from Gaussian multiplicative chaos yield an integrable product of annular weights. A uniform lower mass bound for metric balls then turns that integral into a Hausdorff cover. The graph construction produces one target metric on the whole space; the covering argument is subsequently localized to compact sets. The general-γ sphere assertion is a consequence of this argument, not a theorem quoted from [12], which treats the Brownian sphere.

2 Filled balls and annular paths

We first describe the deterministic geometry. Let D be a proper geodesic metric on $\mathbb{R}^d$ inducing the Euclidean topology. Properness for the random metrics follows from the annular estimates: outward crossings have size at least $\exp((\xi Q - o(1))n)$ at Euclidean radius e^n, by Gaussian tails and the negative-moment bound. Their sum diverges. Closed bounded metric balls are consequently Euclidean bounded and compact. A proper length metric is geodesic, by compactness of parametrized approximate minimizers.

Let $H_r^\circ(x)$ be the complement of the unbounded component of $\mathbb{R}^d \setminus B_r^D(x)$. Let $H_r^\bullet(x)$ be the complement of the unbounded component of $\mathbb{R}^d \setminus \overline{B_r^D(x)}$. The first is open and the second compact. Filling is always toward infinity.

Lemma 2.1 (Hull geometry)**.** *Each hull boundary is a connected compact separator of x and infinity, and every point of that boundary has distance r from x. For $r < R$, the open band*

$$A_{r,R}(x) = H_R^\circ(x) \setminus H_r^\bullet(x)$$

is connected. Its internal distance from the inner boundary to each point of the outer boundary is $R - r$.

Proof. Balls are path connected. Filling preserves connectedness, while the unfilled exterior is connected by definition. We use the elementary unicoherence property of S^d, $d \geq 2$: if two connected closed subsets cover S^d, their intersection is connected. One proof is the degree-zero part of the Mayer–Vietoris sequence, since $H_1(S^d) = 0$; approximation by neighborhoods gives the statement for arbitrary compact sets. Apply it to a filled continuum and the closure of its exterior. This proves boundary connectedness. The same fact, or Alexander duality applied to two disjoint nonseparating continua, shows that the complement of the inner and outer complementary continua of $A_{r,R}$ is connected. Since the band is open, it is path connected.

A hull-boundary point cannot have root distance below r, since it would be in the interior of the ball. If its distance were greater than r, a connected neighborhood avoiding the closed ball would lie in a single complementary component and could not meet the hull boundary. Thus its distance is r.

Take a geodesic from x to an outer-boundary point. Its segment between root-distance levels r and R stays in the band. After leaving the inner hull, it cannot return and later leave again, since every exit point has root distance r. The outer hull is handled in the same way. This gives an internal path of length $R - r$. The triangle inequality gives the reverse bound. Boundary points are interpreted in the internal metric completion. □

The middle of a filled band must be measured by an arrival coordinate. A swallowed component can contain points whose distance from the root exceeds the filling radius.

Fix $A = H^\circ_{t+8u}(x) \setminus H^\bullet_t(x)$, with inner boundary I and outer boundary O. For $0 < v < 8u$, take the internal v-neighborhood of I, adjoin the inner complementary continuum, and fill all complementary components not meeting the outer complementary continuum. Denote this open set by G_v and its closed convention by $G_v^\bullet$. Define

$$T_A(z) = \inf\{v : z \in G_v\}\,.$$

Lemma 2.2 (Arrival coordinate)**.** *The function T_A is 1-Lipschitz for the internal metric of A, with boundary values 0 and $8u$. If $T_A(z) \in [2u, 6u]$, every ambient path of length less than u from z stays in A. In particular, ambient and internal u-balls about z agree.*

Proof. A point of ∂G_v has internal distance v from I, by the neighborhood argument in Lemma 2.1. If $T_A(z) < v < w < T_A(z')$, a path from z to z' leaves G_v at internal distance v from I, while z' has internal distance at least w. Its remaining length is at least $w - v$. Let v, w approach the arrival values. The boundary values follow from Lemma 2.1. A path leaving A first reaches I or O and therefore pays at least $2u$ from a point of the specified middle layer. □

Call A *good* if, for every z with $T_A(z) \in [2u, 6u]$, the inner boundary I is in the component of $\mathbb{R}^d \setminus B^D_u(z)$ containing infinity.

Lemma 2.3. *If A is not good, then* $\operatorname{diam}_D O \le 18u$.

Proof. A ball $B^D_u(z) \subset A$ separates I from O. The terminal length-$8u$ portions of radial geodesics to any two points of O both meet the ball. Join the two meeting points through z at cost less than $2u$. The resulting path has length at most $18u$. □

Fix $\alpha, \zeta \in (0,1)$, put $r_j = R_0\alpha^j$ and $u_j = R_0\alpha^{j-1+\zeta}$, and define the measurable event

$$F_*(x,j) = \{\mathrm{Leb}\,\{v \in [3r_{j-1}/16, r_{j-1}/4] : \operatorname{diam}_D \partial H^\circ_v(x) > 32u_j\} > 0\}\,. \tag{2.1}$$

Let $\widetilde{F}(x,j)$ mean that there is a good band of width $8u_j$, with some center, whose inner hull contains $H^\circ_{r_{j-1}/16}(x)$ and whose outer hull is contained in $H^\circ_{5r_{j-1}/16}(x)$. For sufficiently small α, Lemma 2.3 and hull inclusion give

$$F_*(x,j) \implies \widetilde{F}(y,j) \quad \text{whenever } D(x,y) \le r_{j-1}/16\,. \tag{2.2}$$

Indeed choose a radius v in (2.1), use the band with $t = v - 8u_j$, and note $u_j \le r_{j-1}/128$.

2.1 Shell estimates

Across and connection costs give two kinds of deterministic paths. First, short crossings of successive overlapping Euclidean shells can be joined into a path from the root outward. Second, a crossing P of a closed shell, of length L, together with a connection bound U, gives a separator of controlled diameter. In a fixed larger closed shell take the points at internal distance at most $U + \varepsilon$ from P. Restrict to the closed shell crossed by every competing crossing. This is compact, every crossing meets it, and its internal diameter in the larger buffer is at most $L + 2U + 2\varepsilon$. This use of a closed shell is needed for compactness.

Consequently there are positive local shell marks C_k, L_k, with logarithmic centers $k+\Theta$, where Θ is uniform on $[0,1)$, such that

$$\sum_{j\le k-1} L_j \le D(0,\partial B^E_{e^{k+\Theta}}), \qquad B^E_{e^{k+\Theta+1}} \subset H^\circ_{\sum_{j\le k} C_j}(0)\,. \tag{2.3}$$

All normalized positive moments of C_k and negative moments of L_k are finite. Slight enlargements of the shells make all endpoint and compactness margins strict.

Suppose the radial size at physical logarithmic time t is $Z(t) = e^{\xi W_t}$. On $[k+\Theta, k+\Theta+1)$ set $x(t) = C_{k+1}/Z(t)$ and $y(t) = L_{k-1}/Z(t)$. These are local marks. Define

$$A(t) = \int_{-\infty}^{t} Z(v)y(v)\,dv, \qquad S(t) = \int_{-\infty}^{t} Z(v)x(v)\,dv\,. \tag{2.4}$$

The placement of the marks gives, at every real t,

$$A(t) \le D(0,\partial B^E_{e^t}), \qquad B^E_{e^t} \subset H^\circ_{S(t)}(0)\,. \tag{2.5}$$

The upper integral already includes all shell costs needed to enclose the next Euclidean sphere. The lower integral includes complete inner crossings and at most one fractional crossing that every path to $\partial B^E_{e^t}$ must make.

There are also connected shell paths P_k, with support in $\{e^{t_k-b} < |z| < e^{t_k+b}\}$, length at most U_k, and local lower marks ℓ_k such that every compact separator of the root and infinity meeting this shell has D-diameter at least ℓ_k. For the last assertion, a separator K has $0 \in \operatorname{conv} K$. For $z \in K$ there is therefore $w \in K$ with $|z-w| \ge |z|$. A finite family of small shells around possible z provides a lower crossing that any path from z to w must traverse. Their minimum is ℓ_k.

The global hull-arrival function $\tau(z) = \inf\{r : z \in H^\circ_r(0)\}$ is 1-Lipschitz, by the proof of Lemma 2.2. The intervals $\tau(P_k)$ cover every bounded interval of positive arrival radii. Thus, if

$$\operatorname{diam}_D \partial H^\circ_v(0) \le \Delta \quad \text{for almost every } v \in [b_1R, b_2R],$$

then

$$(b_2-b_1)R \le \sum_{k:\,\mathrm{Leb}(\tau(P_k)\cap(b_1R,b_2R))>0} U_k \mathbf{1}_{\{\ell_k\le\Delta\}}\,. \tag{2.6}$$

Indeed $\tau(P_k)$ has length at most U_k; whenever it contributes positive length it meets a nonexceptional boundary, forcing $\ell_k \le \Delta$. The restriction on the sum is essential at criticality.

3 A stationary radial parameter

A scalar Gaussian component can be separated from the field in every dimension. We use this separation to control annular marks at random physical scales. It also supplies a stationary probability law when the usual critical Brownian exponential integral is infinite.

Choose a nonnegative radial $K \in C^\infty_c(\mathbb{R}^d)$ with $\int K^2 = 1$, and let $k = K * K$. With white noise $\mathcal{W}$ on $\mathbb{R}^d \times (0,\infty)$, define the pinned field

$$h(x) = \int_0^\infty \int_{\mathbb{R}^d} r^{-(d+1)/2} \left[K\!\left(\frac{x-y}{r}\right) - \mathbf{1}_{\{r\ge 1\}} K\!\left(\frac{-y}{r}\right)\right] \mathcal{W}(dy,dr)\,. \tag{3.1}$$

The formula is interpreted on test functions. On tests of total mass zero its covariance is $-\log|x-y|$: the unpinned covariance is $\int_0^\infty k((x-y)/r)\,dr/r$, whose dependence on $|x-y|$ is $-\log|x-y|$ up to a constant.

Project the spatial noise at scale $r=e^t$ onto $K(-\cdot/r)$. With a choice of sign this yields a two-sided Brownian motion B, with $B_0=0$, and an independent Gaussian distribution G such that $h=P(B)+G$, where

$$P(B)(x) = -\int_{\mathbb{R}} [k(e^{-t}x) - \mathbf{1}_{\{t\geq 0\}}]\,dB_t, \tag{3.2}$$
$$\mathrm{Cov}(G(x),G(y)) = \int_{\mathbb{R}} [k(e^{-t}(x-y)) - k(e^{-t}x)k(e^{-t}y)]\,dt.$$

The distribution $G(e^t\cdot)$ is stationary in t. Put $c_K=\int_{\mathbb{R}}[k(e^{-t}e_1)-\mathbf{1}_{\{t\geq 0\}}]\,dt$. Then

$$P(B)(e^t x) - B_t = P(B_{t+\cdot}-B_t)(x), \qquad P(av)(x) = a\log|x| - ac_K\,. \tag{3.3}$$

The latter notation means applying P to the deterministic path $v\mapsto av$.

Lemma 3.1 (Separated Gaussian restrictions)**.** *On any fixed compact annulus away from zero, the maximal Gaussian correlation between the restrictions of $G(e^t\cdot)$ and $G(e^{t+H}\cdot)$ is at most Ce^{-cH}. Consequently, for sufficiently separated logarithmic annuli, nonnegative local functionals Z_i satisfy*

$$\mathbb{E}\prod_i Z_i \leq \prod_i (\mathbb{E}Z_i^{p_0})^{1/p_0} \tag{3.4}$$

with one finite p_0 independent of the number of annuli. The assertion also holds for fixed-width scale windows.

Proof. Let A be the annulus and choose r_0 so small that convolution of a test supported in A with $K(\cdot/r)$ has support disjoint from $K(-\cdot/r)$ for $r<r_0$. Projection removes none of this fine noise. Its Fourier multiplier is

$$m(\omega) = \int_0^{r_0} r^{d-1}|\widehat{K}(r\omega)|^2\,dr \asymp (1+|\omega|^2)^{-d/2}\,.$$

The lower bound uses $\widehat{K}(0)>0$; the upper bound uses rapid Fourier decay. Hence $\mathrm{Var}(G(\nu))\geq c\|\nu\|^2_{H^{-d/2}}$ for tests supported in A.

After rescaling, the cross-covariance kernel for two annuli is

$$\int_{\mathbb{R}} \big[k(e^{-v}(e^{-H}x-y)) - k(e^{-H-v}x)k(e^{-v}y)\big]\;dv\,.$$

The integrand vanishes below a fixed v and the two terms agree at $e^{-H}=0$. Taylor expansion, with integrable bounds for all derivatives, gives a $C^m(A\times A)$ norm at most $C_m e^{-H}$ for every fixed m. Sobolev duality and the preceding lower variance bound give the maximal-correlation estimate. Enlarging A handles fixed-width windows.

For completeness, the Gaussian product inequality follows from covariance domination. If a finite block Gaussian vector has covariance Σ and $\Sigma\leq p_0\,\mathrm{diag}(\Sigma_{ii})$, set

$$u_i(t,x) = \big(P_{1-t}^{\Sigma_{ii}} f_i^{p_0}(x)\big)^{1/p_0},$$

where P_t^A is the Gaussian heat semigroup with covariance tA. Along a Brownian vector of covariance Σ, Itô's formula shows that the drift of $\prod_i u_i(t, B_i(t))$, divided by that product, is

$$\tfrac{1}{2}\left(v^{\mathsf{T}}\Sigma v - p_0 \sum_i v_i^{\mathsf{T}}\Sigma_{ii} v_i\right) \le 0, \qquad v_i = \nabla \log u_i\,.$$

Expectations at $t = 0, 1$ give (3.4). Approximation removes smoothness and positivity restrictions. Summable off-diagonal correlation norms give the required covariance domination. Finite-dimensional approximation proves the field version. A random independent shell phase is first conditioned on; the constants are uniform in that phase. □

Proposition 3.2 (Stationary radial law)**.** *Let Y be the two-sided stationary diffusion with generator and invariant probability measure*

$$\begin{aligned} \mathcal{L}_a f(y) &= \frac{\xi}{2} y f''(y) + (a + \xi/2 - \xi y) f'(y), \\ \pi_a(dy) &= \frac{2^{1+2a/\xi}}{\Gamma(1 + 2a/\xi)} y^{2a/\xi} e^{-2y}\, dy. \end{aligned} \tag{3.5}$$

Independently take G from (3.2). Define

$$t(u) = \frac{1}{\xi}\int_0^u \frac{dv}{Y_v}, \qquad W_{t(u)} = u + \frac{1}{\xi}\log\frac{Y_u}{Y_0}, \qquad H = P(W) + G - Q\log|\cdot|\,. \tag{3.6}$$

Then t is an increasing bijection of $\mathbb{R}$. With $J(t) = \int_{-\infty}^t e^{\xi W_v}\, dv$ and $F = H - \xi^{-1}\log J(0)$,

$$J(t(u)) = \frac{e^{\xi u}}{\xi^2 Y_0}, \qquad \frac{e^{\xi W_{t(u)}}}{J(t(u))} = \xi^2 Y_u\,. \tag{3.7}$$

The law of the full field F, including an independent uniform shell phase, is stationary under

$$F \longmapsto F(e^{t(u)}\cdot) + Qt(u) - u\,. \tag{3.8}$$

Forward from each deterministic u-origin, the physical process W is Brownian motion with drift a, and the corresponding physical arrival times are stopping times in the radial filtration containing the initial state Y_0. In that filtration, $W_t - at$ is Brownian motion for $t \ge 0$.

Proof. The diffusion satisfies $dY = (a + \xi/2 - \xi Y)\, du + \sqrt{\xi Y}\, d\beta$. The zero-current equation gives π_a, and the generator is symmetric in $L^2(\pi_a)$. The usual squared-Bessel construction gives a nonexplosive stationary solution; its Bessel dimension is $2 + 4a/\xi \ge 2$, so it does not hit zero. Ergodicity applied to $\min(1, Y^{-1})$ makes t diverge in both directions.

Set $q(t(u)) = \xi^2 Y_u$. The diffusion time-change formula gives

$$dq = \xi q\, dB + [(\xi a + \xi^2/2)q - q^2]\, dt, \qquad d\log q = \xi\, dB + (\xi a - q)\, dt\,.$$

Differentiating the proposed expression for J in (3.7) gives $dJ/dt = e^{\xi W_t}$ and $J \to 0$ at the negative end. Since $\xi W = \log q + \log J$, this also gives $dW = dB + a\, dt$. The inverse of the adapted, strictly increasing integral J gives the stopping-time assertion.

Shift Y by u_0. Physical time shifts by $t(u_0)$, and $W_{t(u_0)} - u_0 = \xi^{-1}\log(Y_{u_0}/Y_0)$. Formula (3.3) proves the identity in (3.8). Conditionally on the entire radial path, the shift of G is deterministic; its stationarity and independence preserve its law. The independent uniform phase changes

from Θ to $(\Theta - t(u_0)) \bmod 1$, which preserves its conditional law. On punctured compact sets the projection is a smooth function. The negative-time representation proved in Section 7, together with the positive-drift Brownian representation when $a > 0$, gives at most linear growth of W_t in $|t|$; this suffices for the distributional extension through zero. □

We denote this full stationary law by $\mathbb{P}_*$. For $a > 0$ it is the speed-biased law of the usual rooted Gaussian field. For $a = 0$ it is a probability law defined directly by (3.5); no divergent Brownian integral is used.

Proposition 3.3 (Exponential decay)**.** *For $p > 0$,*

$$\left\| e^{U(\mathcal{L}_a - p/(\xi y))} \right\|_{2\to 2} = e^{-\lambda_a(p)U}, \qquad \lambda_a(p) = \sqrt{a^2 + 2p} - a\,. \tag{3.9}$$

For deterministic pairwise disjoint intervals I_i,

$$\mathbb{E}_* \exp\left(-p \sum_i [t(\sup I_i) - t(\inf I_i)] \right) \le \exp\left(-\lambda_a(p) \sum_i |I_i| \right). \tag{3.10}$$

Proof. Put $\nu = \lambda_a(p)/\xi$ and $\psi(y) = y^\nu$. Direct differentiation gives $(\mathcal{L}_a - p/(\xi y))\psi = -\lambda_a(p)\psi$. The Dirichlet form is $\mathcal{E}_a(f,f) = (\xi/2)\int y(f')^2\, d\pi_a$. Integration by parts gives

$$\mathcal{E}_a(f,f) + \frac{p}{\xi}\int y^{-1} f^2\, d\pi_a - \lambda_a(p)\int f^2\, d\pi_a = \frac{\xi}{2}\int y\psi^2\big((f/\psi)'\big)^2\, d\pi_a \ge 0\,. \tag{3.11}$$

Near zero, the derivative and potential terms for ψ have exponent $2\nu + 2a/\xi - 1 > -1$. At infinity the exponential density is integrable. Cutoffs therefore put ψ in the form domain and show that the lower spectral bound is attained. Monotone truncation identifies the nonnegative-potential Feynman–Kac semigroup. To obtain (3.10), compose the killed semigroups on the specified intervals and ordinary norm-one Markov semigroups between them, and take the $L^2(\pi_a)$ inner product with 1 at both ends. □

4 Annular marks at random scales

The exponential estimate applies to deterministic intervals in u. Geometry selects physical scales depending on the field. We control that selection by estimates for entire annular windows and by keeping the relevant arrival intervals in (2.6).

Lemma 4.1 (Physical durations)**.** *For some $b > 0$, if deterministic intervals I_i have lengths at most $c_0 A_i$, where $A_i \ge A_0$, and their enlarged intervals with buffers $c_1 A_i$ are disjoint, then*

$$\mathbb{E}_* \prod_{i=1}^m (1 + t(\sup I_i) - t(\inf I_i))^b \le C^m \prod_{i=1}^m (1 + A_i)^{2b}\,. \tag{4.1}$$

If disjoint intervals have lengths proportional to A_i, the probability that each physical duration is less than a fixed D is at most

$$\exp\left(-c\sum_i A_i^2 + C\sum_i A_i \right). \tag{4.2}$$

Proof. First take $a = 0$ and $T_U = \xi^{-1} \int_0^U Y_v^{-1}\, dv$. Retaining the ground-state component in (3.9) gives

$$\mathbb{E}_* e^{-pT_U} \geq e^{-\sqrt{2p}U} \frac{\Gamma(1+\sqrt{2p}/\xi)^2}{\Gamma(1+2\sqrt{2p}/\xi)}\,.$$

The Gamma quotient is $1 + O(p)$ near zero. The fractional-moment identity

$$\mathbb{E}_* T_U^b = \frac{b}{\Gamma(1-b)} \int_0^\infty (1 - \mathbb{E}_* e^{-pT_U}) p^{-b-1}\, dp$$

then gives $\mathbb{E}_* T_U^b \leq C_b(1+U)^{2b}$ for $b < 1/2$.

At $a = 0$, write $Y = Z_1^2 + Z_2^2$, where independent stationary Ornstein–Uhlenbeck processes satisfy $dZ_i = -(\xi/2)Z_i\, du + (\sqrt{\xi}/2)\, dB_i$. Their variances are $1/4$. The maximal Gaussian correlation of complete path restrictions separated by length H is at most $e^{-\xi H/2}$, by the Gaussian Markov property. Apply (3.4) to the separated intervals and choose b with $bp_0 < 1/2$. This proves (4.1).

For general a, take independently Y_0 with immigration $\xi/2$ and a stationary CIR process Z_a with immigration a, common reversion ξ, and diffusion coefficient $\sqrt{\xi Z_a}$. The sum has generator $\mathcal{L}_a$: its martingale quadratic variations add, and its stationary Gamma shapes add. Consequently $Y_a^{-1} \leq Y_0^{-1}$ in this coupling, proving the same bound. At $a = 0$ take $Z_a = 0$.

Finally, an interval of length ℓ and physical duration at most D satisfies $\int_I Y\, du \geq \ell^2/(\xi D)$ by Cauchy–Schwarz. For $0 < v < \xi/2$, put $\kappa = 1 - \sqrt{1 - 2v/\xi} < 1$. The function $e^{\kappa y}$ is a ground state of $\mathcal{L}_a + vy$ with eigenvalue $(a + \xi/2)\kappa$. The form argument in (3.11), now with this exponential ground state, bounds the semigroup norm by $e^{(a+\xi/2)\kappa U}$. For the unbounded positive potential, first truncate vy; the same form bound is uniform in the truncation. Monotone convergence in the Feynman–Kac formula gives the stated semigroup estimate. Composition over disjoint intervals and exponential Markov inequality give (4.2). □

A local mark $M(t) \geq 1$ below dominates a fixed finite collection of normalized crossing costs, inverse crossing costs, connection costs, and their fixed-width suprema. The normalization uses the current radial value W_t. All its supporting shells, including a preceding lower crossing, lie in one fixed enlarged annulus. Choose $\operatorname{supp} k$ smaller than the inner radius of that annulus. The smooth radial component of the normalized field on it is

$$R_t(x) = \int_0^\infty [1 - k(e^{-v}x)]\, dW_{t+v}\,. \tag{4.3}$$

The kernel and its spatial derivatives decay exponentially as $v \to \infty$. Its negative-time part is zero because of the support choice. The deterministic drift a contributes a bounded smooth function.

Lemma 4.2 (Adaptive radial estimates)**.** *For sufficiently separated nonnegative stopping times $S_1 < \cdots < S_m$ of the forward radial filtration,*

$$\mathbb{E}_* \exp\left(c \sum_{i=1}^m \left(\sup_{0 \leq v \leq 1} \|R_{S_i+v}\|_A \right)^2 \right) \leq C^m\,. \tag{4.4}$$

The same holds with the Brownian oscillation on each cell added inside the individual squares.

Proof. After a spatial cutoff, expand the kernel in a Fourier basis and include Sobolev weights of order greater than $d/2$. Its coefficients satisfy $|f_\ell(v)| \leq Ca_\ell e^{-cv}$, with $\sum_\ell a_\ell < \infty$. The random integrals

$$X_{i\ell} = \int_{S_i}^\infty f_\ell(v - S_i)\, dB_v$$

have predictable integrands. Spacing and exponential kernel decay bound the quadratic variation of every linear combination by $C\sum_{i,\ell} a_\ell \lambda_{i\ell}^2$. The exponential martingale inequality therefore gives

$$\mathbb{E}_* e^{\sum \lambda_{i\ell} X_{i\ell}} \le e^{C\sum a_\ell \lambda_{i\ell}^2/2}.$$

Integrating this inequality with $\lambda_{i\ell} = \sqrt{2c}\, G_{i\ell}$ for independent standard Gaussian variables gives

$$\mathbb{E}_* e^{c\sum_{i,\ell} X_{i\ell}^2} \le \prod_\ell (1-2cCa_\ell)^{-m/2} \le C_1^m$$

for small c. Sobolev embedding controls the spatial supremum. The identity

$$R_{S+v}(x) = R_S(e^v x) - (W_{S+v} - W_S)$$

controls the time supremum after enlarging the annulus. Strong Markov and the deterministic cell spacing give the product exponential bound for the Brownian oscillations. Hölder's inequality completes the proof. Adding the bounded drift changes C only. □

If candidate times have the required spacing only on an event, replace them recursively by $S_1' = S_1$, $S_i' = \max(S_i, S_{i-1}' + D)$. These are stopping times, always separated, and agree with the candidates on that event. No conditioning on a future spacing event is needed.

The residual mark $M_S^\perp$, depending only on G in an enlarged annulus, has all positive moments. To see this, add an independent Gaussian radial projection to recover an ordinary local field. Weyl comparison bounds the residual mark by the ordinary mark times an exponential of a smooth Gaussian supremum. Hölder and Proposition 1.1 apply. On a physical unit cell,

$$\sup_{0\le v\le 1} M(S+v) \le C M_S^\perp \exp\left(C \sup_{0\le v\le 1} \|R_{S+v}\|_A + C \operatorname{osc}_{[S,S+1]} W \right). \tag{4.5}$$

This also proves every positive one-time moment under $\mathbb{P}_*$, uniformly in u by stationarity.

Lemma 4.3 (Marks on buffered intervals)**.** *Fix $c_0, c_1, v > 0$. Let $I_i = [b_i, c_i]$ be deterministic, with $|I_i| \le c_0 A_i$, $A_i \ge A_0$, and suppose $[b_i - c_1 A_i, c_i + c_1 A_i]$ are disjoint. For every $B > 0$, after increasing A_0,*

$$\mathbb{P}_*\left[\sup_{u\in I_i} M(t(u)) > e^{vA_i} \text{ for every } i \right] \le C^m e^{-B\sum_i A_i}. \tag{4.6}$$

Proof. Shift the earliest enlarged left endpoint to zero. All physical times are then forward stopping times. Call a right buffer good if its physical duration exceeds $D+2$, with D larger than all support and stopping-cell widths. Formula (4.2) bounds simultaneous bad buffers by $\exp(-c\sum A_i^2 + C\sum A_i)$.

Choose a large L. By (4.1) and Markov's inequality, any specified set of cores with physical duration greater than e^{LA_i} has probability at most $\prod C(1+A_i)^{2b} e^{-bLA_i}$. On the remaining event, core i has at most Ce^{LA_i} physical unit cells, with stopping-time anchors $\min(t(b_i)+k, t(c_i))$.

Good buffers separate the anchors from different cores. Lemma 4.2, with delayed anchors off the spacing event, and a union over the cells bound a radial exceedance in every selected core by $C^m \exp(L\sum A_i - c\sum A_i^2)$. For the residual term condition on the complete radial path and phase. The cell locations are now deterministic, and Lemma 3.1 and arbitrary residual moments bound their simultaneous exceedances by $C_q^m \exp(-(qv/C - L)\sum A_i)$.

An excessive mark has one of these four causes: a short buffer, a long core, a radial exceedance, or a residual exceedance. Partition the indices by a selected cause. Hölder applied to the four events loses at most a factor four in the exponent. Choose L, then q, then A_0 large enough, and sum over the 4^m partitions. This gives (4.6). □

Proposition 4.4 (Simultaneous envelope bounds)**.** *Set*

$$\mathcal{R}(u) = \xi \int_{-\infty}^{u} e^{-\xi(u-v)} M(t(v))\, dv\,. \tag{4.7}$$

For every $\varepsilon, \delta, B > 0$, there are L_0, C such that, for $L \geq L_0$, $n \geq 1$, and any deterministic v_0,

$$\mathbb{P}_*[\#\{1 \leq j \leq n : \mathcal{R}(v_0 + jL) > e^{\varepsilon L}\} \geq \delta n] \leq Ce^{-BnL}\,. \tag{4.8}$$

Moreover $S(t(u))/J(t(u))$ and $J(t(u))/A(t(u))$ are bounded by $\mathcal{R}(u)$ after enlarging M.

Proof. Jensen and stationarity give every positive moment of $\mathcal{R}(0)$. Set $v_0 = 0$. The contribution from $v < 0$ at jL is $e^{-\xi jL}\mathcal{R}(0)$. If it accounts for $\delta n/2$ exceedances, then $\mathcal{R}(0) \geq ce^{c\delta nL}$; arbitrary moments give the required rate.

Divide $[0, nL]$ into cells of length $\varepsilon' L$, with ε' fixed and small. For a cell I_k put $T_k = \log(1 + \sup_{v \in I_k} M(t(v)))$. Any remaining exceedance at jL has a preceding witness with

$$T_k \geq \varepsilon L/2 + (\xi/2)\operatorname{dist}(jL, I_k)\,;$$

otherwise summing the exponential kernel gives at most $Ce^{\varepsilon L/2} < e^{\varepsilon L}/2$. A witness influences only a forward interval of length $C(T_k + L)$. Enlarge these intervals symmetrically by a fixed factor and select a disjoint subfamily whose triples cover their union. Such a family is obtained by choosing a longest interval, deleting those meeting it, and repeating. Each deleted interval lies in the triple of its selected interval.

All relevant heights satisfy $T_k \geq \varepsilon L/2$. An interval of length CT_k contains at most $C_\varepsilon T_k/L$ of the grid points. Thus the selected family satisfies $\sum T_k \geq c_{\varepsilon,\delta} nL$. Round its heights down to $A_k = a_0 L m_k$, where $m_k \geq 1$ is integral and a_0 is small. Then $A_k \leq T_k < 2A_k$, the original cells have length $O(A_k)$, and their buffers of size $c_1 A_k$ fit in the selected disjoint intervals.

Lemma 4.3 bounds each specified family and height assignment by $C^m e^{-B' a_0 L \sum m_k}$. There are $O(n)$ cell centers, and at most C^{n+M} choices with $M = \sum m_k$; this follows by choosing the centers and a composition of M. Also $M \geq cn$. Choose B' and then L large and sum over M. This proves (4.8), including arbitrarily large individual heights.

Finally $e^{\xi W_t}\, dt = dJ = \xi J\, du$. Comparing the integrands in (2.4) gives $S/J \leq \mathcal{R}$. Jensen for $z \mapsto z^{-1}$ gives $J/A \leq \mathcal{R}$ when $M \geq y^{-1}$. □

Lemma 4.5 (Occupation near zero)**.** *For $0 < r < 1/4$ and every f in the Dirichlet-form domain,*

$$\int_0^r f^2\, d\pi_a \leq Cr(1 + \log(1/r))\big(\mathcal{E}_a(f,f) + \|f\|_2^2\big)\,. \tag{4.9}$$

If $vCr(1 + \log(1/r)) =: \theta < 1$, then for deterministic disjoint intervals I_i,

$$\mathbb{E}_* \exp\left(v \sum_i \int_{I_i} \mathbf{1}_{\{Y_u \leq r\}}\, du\right) \leq e^{\theta \sum_i |I_i|}\,. \tag{4.10}$$

Proof. Put $b = 2a/\xi$. For $y < 1/2$ and $z \in [1/2, 1]$, weighted Cauchy–Schwarz gives

$$|f(y)|^2 \leq 2|f(z)|^2 + 2\left(\int_y^z s^{-b-1}\, ds\right)\left(\int_y^z s^{b+1}|f'(s)|^2\, ds\right)\,.$$

Average over z and integrate over $y \in (0,r)$ against $y^b dy$. The energy coefficient is

$$\int_0^r y^b \int_y^1 s^{-b-1}\, ds\, dy = \begin{cases} r(1+\log(1/r)), & b=0, \\ [r - r^{b+1}/(b+1)]/b, & b>0. \end{cases}$$

For fixed b it is bounded by a constant times $r(1+\log(1/r))$. The exponential density factors are bounded on $(0,1)$, which proves (4.9). The form of $\mathcal{L}_a + v\mathbf{1}_{(0,r]}$ is at most $-(1-\theta)\mathcal{E}_a + \theta\|\cdot\|_2^2$. Its semigroup has norm at most $e^{\theta U}$. Composing over the disjoint intervals proves (4.10). □

Proposition 4.6 (Absence of many thin bands)**.** *Fix $0 < b_1 < b_2 < \infty$, $C_0, \zeta > 0$, and $L_f < \infty$. In the stationary rooted metric, allow any possibly field-dependent continuous perturbation f with $\|f\|_\infty \le L_f$. Put $R_j = R_0\alpha^j$ and $\Delta_j = C_0\alpha^\zeta R_j$. If E_j^f is the event*

$$\operatorname{diam}_{D_{F+f}} \partial H_v^\circ(0) \le \Delta_j \quad \text{for almost every } v \in [b_1R_j, b_2R_j],$$

then, for every $\delta, B > 0$ and sufficiently small α,

$$\mathbb{P}_*[\#\{0 \le j < n : E_j^f\} \ge \delta n] \le C\alpha^{Bn}. \tag{4.11}$$

The bound is uniform in deterministic $R_0 > 0$ and holds for one measurable base-environment obstruction containing the events for all such perturbations.

Proof. First take $f = 0$. Absorb $J(0)$ into W, so that $J(t(u)) = e^{\xi u}$ and $e^{\xi W_{t(u)}} = \xi^2 Y_u J(t(u))$. Write $K_0 = \log(1/\alpha)$ and choose $\varepsilon > 0$ with $6\varepsilon < 1$ and $\varepsilon < \zeta/12$. For $R = e^{\xi u_0}$ put

$$u_\pm = u_0 \pm 2\varepsilon K_0/\xi, \qquad u_{\pm\pm} = u_0 \pm 3\varepsilon K_0/\xi, \qquad I_R = [u_{--}, u_{++}].$$

Suppose $S(t(u_-))/J(t(u_-))$ and $J(t(u_+))/A(t(u_+))$ are at most $e^{\varepsilon K_0}$. Then $S(t(u_-)) < b_1R$ and $A(t(u_+)) > b_2R$ for large K_0. By (2.5), a shell contributing to (2.6) has center in $[t(u_-) - b, t(u_+) + b]$. Require the two padding physical durations to exceed $b + H + 1$, where H is the fixed shell spacing. All contributing shell cells then lie inside the physical image of I_R, and

$$Re^{-3\varepsilon K_0} \le J(t(u)) \le Re^{3\varepsilon K_0} \qquad (u \in I_R).$$

Enlarge the local mark so that on a shell cell $[t_k, t_k + H]$,

$$U_k \le C\int_{t_k}^{t_k+H} e^{\xi W_t} M(t)\, dt, \qquad \ell_k \ge e^{\xi W_t}/M(t).$$

If also $\sup_{u\in I_R} M(t(u)) \le e^{\varepsilon K_0}$, the condition $\ell_k \le C_0 Re^{-\zeta K_0}$ forces

$$Y_u \le r_{K_0} := Ce^{-(\zeta-4\varepsilon)K_0}$$

throughout the image of that cell. Since $e^{\xi W_t}dt = \xi J(t(u))du$, the thin-band inequality (2.6) now implies

$$\int_{I_R} \mathbf{1}_{\{Y_u \le r_{K_0}\}}\, du \ge ce^{-4\varepsilon K_0}. \tag{4.12}$$

The intervals I_{R_j} are deterministic, disjoint, and have length $6\varepsilon K_0/\xi$. Apply Lemma 4.5 with $v = r_{K_0}^{-1/2}$. Its spectral error is $O(\sqrt{r_{K_0}}(1+\log(1/r_{K_0})))$. Exponential Markov inequality bounds simultaneous occurrences of (4.12) on any specified m intervals by

$$\exp\big(-cme^{(\zeta/2-6\varepsilon)K_0}\big).$$

The exponent $\zeta/2 - 6\varepsilon$ is positive.

Failure of an envelope condition at a positive fraction of the scales has any prescribed e^{-BnK_0} rate by Proposition 4.4. Failure of a padding duration has the same bound by (4.2). Failure of the mark bound has it by Lemma 4.3, with $A_i = K_0$ and buffers fitting between the I_{R_j}. A positive fraction of thin events therefore forces a positive fraction of one of these failures or of (4.12). Sum over at most 2^n specified subcollections to obtain (4.11).

For general f, set $\Lambda = e^{\xi L_f}$. All shell costs increase by at most Λ, all separator lower bounds decrease by at most Λ^{-1}, and the hull envelopes hold with ΛS and $\Lambda^{-1}A$. These deterministic changes are absorbed by increasing K_0. They give a common obstruction in the original environment, so the argument also covers random f. □

5 Products of geometric weights

The preceding estimates allow us to use the exact decay rate without conditioning the radial process on a randomly selected set of scales. For a bounded hull centered at x, write $\operatorname{out}_x H = \sup_{z\in H} |z-x|$ and $\operatorname{in}_x H = \sup\{r : B^E_r(x) \subset H\}$.

Proposition 5.1 (Rooted product estimate)**.** *Fix $p > d$ and $L_f < \infty$. There are $\alpha, \zeta \in (0,1)$, $\eta \in (0, 1/2)$, and $q > s$ such that, for $r_j = R_0\alpha^j$, $u_j = R_0\alpha^{j-1+\zeta}$, the factors*

$$v_j^f = \left(\eta + 128 \frac{\operatorname{out}_0 H^\bullet_{64r_j}(0)}{\operatorname{in}_0 H^\circ_{u_j/2}(0)} + 21_{F^f_*(0,j)^c} \right) \wedge 1 \tag{5.1}$$

satisfy

$$\mathbb{E}_* \left[\left(\prod_{j=1}^n v_j^f \right)^p \right] \le C\alpha^{qn} . \tag{5.2}$$

All hulls use D_{F+f}. The estimate is uniform in deterministic $R_0 > 0$ and all continuous f with $\|f\|_\infty \le L_f$, including random f. One common choice of parameters works for any prescribed finite set of changes to the numerical radius multipliers, ratio coefficient, and boundary-diameter threshold. The parameter η can be chosen as α^{M_0} with arbitrarily large M_0.

Proof. Put $\beta(p) = \lambda_a(p)/\xi > s$. Choose small $\delta, \zeta, \varepsilon > 0$ such that

$$\vartheta = (1-\delta)(1-\zeta-4\varepsilon)\beta(p) > s, \qquad s < q < \vartheta .$$

Set $\eta = \alpha^{M_0}$ with $M_0 p > \beta(p) + 1$ and $M_0 > 2$, and then choose $K_0 = \log(1/\alpha)$ large. Introduce deterministic intervals

$$I_j = \left[\frac{\log(64r_j) + 2\varepsilon K_0}{\xi}, \frac{\log(u_j/2) - 2\varepsilon K_0}{\xi} \right] = [u_j^-, u_j^+] .$$

They are disjoint and have common length $\ell = ((1-\zeta-4\varepsilon)K_0 - \log 128)/\xi$.

For the normalized field F, use $A_F = A/J(0)$ and $S_F = S/J(0)$, so that $J_F(t(u)) = e^{\xi u}$. By Propositions 4.4 and 4.6, except on an event of probability $C\alpha^{Bn}$ for arbitrarily large fixed B, at least $(1-\delta)n$ indices satisfy F^f_* and both envelope bounds at both endpoints. At such an index Weyl comparison and $e^{\varepsilon K_0} > e^{\xi L_f}$ imply

$$H^\bullet_{64r_j}(0) \subset B^E_{e^{t(u_j^-)}}(0), \qquad B^E_{e^{t(u_j^+)}}(0) \subset H^\circ_{u_j/2}(0) .$$

Thus, with $\Delta_j = t(u_j^+) - t(u_j^-)$,

$$(v_j^f)^p \le 2^{p-1}(\eta^p + 128^p e^{-p\Delta_j}) .$$

For a deterministic set G of indices, expand the product of these upper bounds and apply (3.10) to each term. Its expectation is at most $[2^{p-1}(\eta^p + 128^p e^{-\lambda_a(p)\ell})]^{|G|}$.

On the event with $m = \lceil (1-\delta)n \rceil$ good indices, discard all other factors and sum over the at most 2^n deterministic choices of G of size m. The resulting bound is $C\alpha^{Bn} + C_1^n \alpha^{\vartheta n}$. Choose $B > q$ and then K_0 so large that $C_1 \alpha^{\vartheta - q} \le 1$. Fixed multiplier changes alter ℓ by bounded constants and leave the strict inequality unchanged. □

5.1 Root completion and measurability

The metric at the singular root used above is defined by its punctured local metrics and completion. The path-density and finite-rooting identities in Section 7 are proved independently of the product estimate and filled-ball geometry; they may therefore be used here. Inward connection decay uses only the local marks and convolution estimate of Proposition 4.4, which are defined on the punctured field. Under ordinary γ-rooting, $a > 0$ gives inward radial behavior $B_t - at$ and summable annular paths. Under truncated critical rooting the inward deficit is a three-dimensional Bessel process. The rooted topology also follows directly from the finite rooting identity: a probability-one continuity property of the unrooted metric remains true at almost every sampled root. Its restriction to the punctured domain is the local Weyl metric, hence determines the completed root.

Here is the identification with the punctured intrinsic completion. For $\varepsilon > 0$, an ordinary small ball around the sampled root has diameter less than ε by continuity. A path through the root can be modified inside a yet smaller annulus: its entering and leaving crossings are joined by a connection mark whose cost tends to zero. Indeed the local connector bound $S/J \le \mathcal{R}$, Proposition 4.4, and $J(t(u)) = J(0)e^{\xi u}$ make the inward network costs tend to zero almost surely, first along integer u and then by inclusion. Any two approaches to the root therefore have vanishing intrinsic distance. There is exactly one added point. The finite-density comparisons in Section 7 transfer this conclusion to $\mathbb{P}_*$. At infinity, forward Brownian motion with drift $a \ge 0$ and the lower envelope show properness; equivalently $A(t(u)) \ge J(t(u))/\mathcal{R}(u)$ diverges along integer $u \to \infty$ by arbitrary moments and Borel–Cantelli.

Only F_* and the ratios in (5.1) are integrated. Their measurability can be checked on the one-point compactification. The continuous function $D(x,z)/(1 + D(x,z))$, with value 1 at infinity, has Borel compact superlevel sets. Taking the component containing infinity is an upper semicontinuous compact-set operation: every Hausdorff limit of such components is connected, contains infinity, and lies in the limiting component. Boundary, complement, and monotone compact-limit operations are Borel, tested against a countable basis. Also

$$H_r^\bullet(x) = \bigcap_{m \ge 1} \overline{H_{r+1/m}^\circ(x)} .$$

Distances, diameters, and Euclidean inner and outer radii of these sets are consequently Borel. Integrating the boundary-diameter indicator in (2.1) preserves measurability. The existential event $\widetilde{F}$ will only be used in pathwise deterministic comparisons.

6 Uniform mass of metric balls

We need a bound valid at every center in a compact set. A statement at a chaos-typical point is not sufficient for the covering argument. The critical case is handled by counting visits of the coarse field near its upper barrier, rather than by a positive power bound on Euclidean ball masses.

6.1 Field and chaos inputs

Let X be the fine part of (3.1), with scales $r < 1$. Write H_t for its scales between e^{-t} and 1. Then

$$\operatorname{Cov}(H_t(x), H_v(y)) = \int_0^{t\wedge v} k(e^u(x-y))\,du\,. \tag{6.1}$$

At fixed x, this is standard Brownian motion in t. The rescaled field $(X - H_n)(z + e^{-n}\cdot)$ has the law of X and is independent of the coarse noise through time n. Locally $h = X + g$ with g smooth, so bounded local Weyl comparison transfers the conclusions to h.

We use the following precise foundational facts about Gaussian multiplicative chaos. For $\gamma < \gamma_c$ let μ_X be the limit of $e^{\gamma H_t - \gamma^2 t/2}dx$. At $\gamma = \gamma_c$ let μ_X be the critical derivative limit of $(\gamma_c t - H_t)e^{\gamma_c H_t - dt}dx$, with its positive barrier approximation. These measures are locally finite, atomless, have full support, and their mass on each fixed nonempty bounded open set has all negative moments. They satisfy continuous Weyl multiplication $\mu_{X+f} = e^{\gamma f}\mu_X$ and the coarse decomposition

$$\mu_X(dx) = e^{\gamma H_n(x) - \gamma^2 n/2}\mu_{X-H_n}(dx)\,. \tag{6.2}$$

At criticality the extra derivative term is a bounded coarse multiplier of ordinary critical chaos, which tends to zero. These facts and the rooting identities used below are the standard construction and moment theorems for subcritical and critical chaos [13, 6, 7]. They are assumed as field-theoretic foundations, not proved anew in this paper.

The other background input is the Gaussian maximum upper tail of [5, Proposition 1.1]. For a box of side N in $\mathbb{Z}^d$, a centered Gaussian field whose variances are all equal and at most $\log N + C$, and whose increment variances are at most $2\log_+|u-v| + C$, has maximum M_N satisfying

$$\mathbb{P}[M_N \ge \gamma_c \log N - \tfrac{3}{2\gamma_c}\log\log N + z] \le C(1+z)e^{-\gamma_c z} \qquad (z \ge 1)\,. \tag{6.3}$$

Its hypotheses hold for H_n on $\kappa e^{-n}\mathbb{Z}^d$ in a fixed box. Indeed its variance is n, and (6.1), $0 \le k \le 1$, and $1 - k(z) = O(|z|^2)$ give the required increment bound after indexing the grid. The grid side is $N \asymp e^n$. Therefore, for every $a_0 < 1/(2\gamma_c)$, the probability that a grid value exceeds $\gamma_c n - a_0 \log n$ is at most $C(\log n)n^{-3/2+\gamma_c a_0}$, which is summable.

Lemma 6.1 (Uniform deficit summability)**.** *Put $V_t(x) = Qt - H_t(x)$, where $Q \ge \gamma_c$. For every compact box J, $\eta > 0$, and $0 < \theta < \eta/(2\gamma_c)$, almost surely*

$$\sup_{x\in J}\sum_{n\ge m} e^{-\eta V_n(x)} \le Cm^{-\theta} \tag{6.4}$$

for all sufficiently large m. The same statement holds with an integral in t.

Proof. It suffices to prove the assertion for $Q = \gamma_c$, since increasing Q increases every deficit. Fix a buffer constant C_* and define

$$R_n(z) = \sup_{0\le t\le n}\ \sup_{|x-z|\le C_* e^{-n}} |H_t(x) - H_t(z)|\,.$$

For $|a| \le Mn$, Gaussian conditioning gives

$$\mathbb{P}[R_n(z) > u \mid H_n(z) = a] \le Ce^{-cu^2} \qquad (u \ge C)\,. \tag{6.5}$$

The conditional mean is bounded because its covariance with $H_n(z)$ is $O(1)$. On the time strip $[n-j-1, n-j]$, the variance and rescaled spatial increment bounds are $O(e^{-2j})$. Gaussian entropy and summation in j bound the expected centered supremum; Gaussian concentration proves (6.5).

By (6.3) and a Gaussian height bound, eventually $V_n(z) \ge a_0 \log n$ and $|H_n(z)| \le Mn$ at every relevant grid point. At $H_n(z) = \gamma_c n - v$, its Gaussian density is

$$(2\pi n)^{-1/2} e^{-dn + \gamma_c v - v^2/(2n)}\,. \tag{6.6}$$

For $1/2 < \rho < 1$, the union bound using (6.5) and $O(e^{dn})$ points is bounded by

$$Cn^{-1/2} \int_{a_0 \log n}^{\infty} e^{\gamma_c v - cv^{2\rho}}\, dv\,.$$

It is summable. Hence $R_n(z) \le V_n(z)^\rho$ eventually, uniformly on the grid. Nearest grid points then imply $V_n(x) \ge a \log n$ uniformly in x, for every $a < 1/(2\gamma_c)$.

We next prove, for each fixed B, that eventually at every dyadic N,

$$\sup_{x \in J} \#\{k \in [N, 2N] \cap \mathbb{Z} : V_k(x) \le B \log N\} \le (\log N)^4\,. \tag{6.7}$$

If a point has such visits, let n be its last visit and choose its nearest point z on the scale-n grid. The oscillation bound gives $V_n(z) \in [a \log N, b \log N]$ and preserves all visits with slightly enlarged constants, uniformly for $k \le n$. Conditional on $V_n(z) = v$, the process $H_k(z) - (k/n)(\gamma_c n - v)$ is a Brownian bridge B_k from zero to zero in time n. The lower deficit bound is the upper constraint $B_k \le kv/n - a \log N$; a counted visit requires $B_k \ge kv/n - b \log N$.

Set $L = \lceil (\log N)^2 \rceil$. At a stopping time $\tau \le n - 2L$ satisfying the lower visit constraint, the conditional variance of $B_{\tau+L}$ is at least $L/2$, and its mean is at least $-C \log N$. The upper constraint at $\tau + L$ is at most $C \log N$. Thus it is violated with probability at least a fixed $p_0 > 0$. Select the first visit, test after L steps, and restart at the first later visit. Apart from the final $2L$ times, every L visits produce another test. The Markov property of the bridge bounds the probability of m visits while respecting the upper constraint by $Ce^{-cm/(\log N)^2}$.

Integrating (6.6), summing over $O(e^{dn})$ grid points and $N+1$ terminal times gives at most $CN^{\gamma_c b + 1/2} (\log N) e^{-c(\log N)^2}$ for $(\log N)^4$ visits. This is summable over dyadic N and proves (6.7).

Choose $a < 1/(2\gamma_c)$ with $\eta a > \theta$, and B with $\eta B > 1 + \theta$. The sum on $[N, 2N]$ is at most

$$(\log N)^4 N^{-\eta a} + (N+1) N^{-\eta B} \le CN^{-\theta}\,.$$

Sum over dyadic blocks. Including $\sup_{n \le t \le n+1} |H_t(x) - H_n(x)|$ in the conditional oscillation argument gives a bound $o(V_n(x))$ uniformly. Apply the discrete assertion with a slightly smaller coefficient than η to obtain the integral version. □

Proposition 6.2 (Ball mass)**.** *For every compact $K \subset \mathbb{R}^d$ and every $q_0 > s = \gamma/\xi$, almost surely there are $c, r_0 > 0$ such that*

$$\mu_h(B_r^{D_h}(x)) \ge cr^{q_0} \qquad (x \in K,\ 0 < r < r_0)\,. \tag{6.8}$$

The assertion persists after any continuous field perturbation, with locally adjusted constants.

Proof. Work first with X. At scale e^{-n} and grid center z, let $L_n(z)$ be a crossing and separator-connection cost in a fixed buffered cube, and $m_n(z) = \mu_X(B^E_{e^{-n}/64}(z))$. The separator can be obtained from the buffered connection construction in Section 2. Coarse independence, Weyl comparison, (6.5), and the negative moments of fine chaos give, for every $b > 0$,

$$\mathbb{E}\left[(e^{\xi V_n(z)}L_n(z))^b + (e^{\gamma V_n(z)}m_n(z))^{-b} \mid H_n(z) = a\right] \le C_b \quad (|a| \le Mn)\,. \tag{6.9}$$

The mass normalization follows from $\gamma Q = d + \gamma^2/2$: after rescaling (6.2), its constant factor is $e^{\gamma H_n(z) - \gamma Q n} = e^{-\gamma V_n(z)}$.

For general $Q \ge \gamma_c$, the density in (6.6) has Q in place of γ_c and factor $e^{-Q^2 n/2}$ in place of e^{-dn}. Multiplication by the grid count leaves $e^{(d-Q^2/2)n} \le 1$. Conditional Markov inequality, the lower barrier, and arbitrary b therefore imply, for every $\varepsilon > 0$, eventually at every relevant grid point,

$$L_n(z) \le e^{-(\xi-\varepsilon)V_n(z)}, \qquad m_n(z) \ge e^{-(\gamma+\varepsilon)V_n(z)}\,. \tag{6.10}$$

For example the exceptional probability is at most $C_b n^{-1/2}\int_{a_0 \log n}^{\infty} e^{-(b\varepsilon - Q)v}\,dv$, which is summable for large b.

Fix $\vartheta < \xi/\gamma$ and choose $\varepsilon, \eta > 0$ with $\vartheta(\gamma+\varepsilon) + \eta < \xi - \varepsilon$. For a small open cube U and a mark whose buffer lies in U, put $V = \mu_X(U)$. Equations (6.10) give

$$L_n(z) \le m_n(z)^{\vartheta} e^{-\eta V_n(z)} \le V^{\vartheta} e^{-\eta V_n(z)}\,. \tag{6.11}$$

We claim $\operatorname{diam}_{D_X}(U;U) \le CV^{\vartheta}$, uniformly for all sufficiently small such cubes in a fixed box.

Let c and ℓ be the center and side of U. Choose a sufficiently small fixed b_0, and m with $e^{-m} \in [b_0\ell/e, b_0\ell]$. For $x \in U$ use interpolation points $y_j = x + e^{-j}(c - x)$ and scales $r_j = e^{-m-j}$. Their distance from ∂U is at least $\ell e^{-j}/2$, and $|y_j - x| \le C_{b_0} r_j$. Round to the grids and connect consecutive rounded points by monotone coordinate grid paths at the larger scale. Choose b_0 and then κ small enough that all buffers remain in U. The number of marks per level is bounded by a fixed constant, and their centers are within $C_* r_j$ of x.

Crossings at neighboring centers, with equal radii or ratio e^{-1}, both cross the larger center's separator: their inner endpoints are well inside its inner cube and their outer endpoints well outside its outer cube. For example use crossing radii $r/100$ and $100r$, and separator radii r and $2r$. Connection costs join them in the buffered cube. The initial central crossing is the same for every x. The oscillation estimate in Lemma 6.1 gives $V_n(z) \ge V_n(x)/2$ for all centers in this construction, eventually. Its total length is therefore at most

$$CV^{\vartheta} \sum_{n \ge m} e^{-(\eta/2)V_n(x)} \le C' V^{\vartheta}\,.$$

The tails converge to x in Euclidean topology and have vanishing metric length, so the path extends to x. Join two such paths through the common central crossing. This proves the claimed internal diameter bound. All random estimates came from one countable grid; the cube and endpoints are chosen pathwise afterward.

Now take $\vartheta = 1/q_0$. For $x \in K$ let $U = x + (-t, t)^d$ be the largest centered cube contained in $B^{D_X}_{r/2}(x)$. More precisely $t = \inf\{|z - x|_\infty : D_X(x, z) \ge r/2\}$. A minimizer y exists in a fixed compact neighborhood and has $D_X(x, y) = r/2$, by continuity along the segment from x to y. Uniform compatibility of the two topologies makes $t \to 0$ uniformly on K. Approximate y from inside U to obtain

$$r/2 \le \operatorname{diam}_{D_X}(U;U) \le C\mu_X(U)^{1/q_0} \le C\mu_X(B^{D_X}_r(x))^{1/q_0}\,.$$

This is (6.8). On a compact neighborhood, continuous Weyl factors compare both metrics and masses by finite constants. All sufficiently small relevant balls and paths remain there, proving the perturbation assertion and transfer to h. □

An additional consequence of Lemma 6.1 is that

$$J_X(x) = \int_0^\infty e^{\xi(H_t(x)-Qt)}\,dt \tag{6.12}$$

is finite, positive, and continuous in x, with uniformly vanishing tails on compact sets. Truncated integrals are continuous, and (6.4) gives uniform convergence. Thus J_X and J_X^{-1} are bounded on every compact set.

7 Removing the sampled root

We now transfer the stationary estimate to an almost sure spatial integral. The two regimes require different rooting laws, but the localization and the covering factors are the same.

7.1 Subcritical rooting

Under ordinary Gaussian γ-rooting at zero, the inward scalar path is $B_t - at$, where $a = Q - \gamma > 0$. Extend it by independent outer noise. Equivalently, in physical logarithmic time take a two-sided $W_t = B_t + at$. Its exponential integral $J(t) = \int_{-\infty}^t e^{\xi W_v}\,dv$ is finite, and $q(t) = e^{\xi W_t}/J(t)$ is stationary under physical scale shifts. Its logistic equation is the one in Proposition 3.2. The stationary density of q/ξ^2 is Gamma with shape $2a/\xi$ and rate 2, so $\mathbb{E}_0 q(0) = \xi a$.

The speed $du/dt = q/\xi$ changes the invariant measure by this speed. To verify it for the entire environment, let θ_t be the stationary physical flow, Ψ_u its reparametrization, and $v = q/\xi$. For bounded f, set $G_u = \int_0^u f \circ \Psi_w\,dw$. The change-of-variables identity

$$G_u(\theta_T\omega) - G_u(\omega) = \int_0^T v(\theta_t\omega)[f(\Psi_u\theta_t\omega) - f(\theta_t\omega)]\,dt$$

shows, after expectation, that $v\,d\mathbb{P}_0/\mathbb{E}_0 v$ is invariant under Ψ. Its radial invariant density is π_a, and its independent residual and phase have the stationary law of Proposition 3.2. Since $W_0 = 0$,

$$\frac{d\mathbb{P}_*}{d\mathbb{P}_0} = \frac{1}{\xi a J(0)}, \qquad \mathbb{E}_0[\mathbf{1}_{\{J(0)\le T\}}\Phi] \le \xi a T \mathbb{E}_*\Phi \tag{7.1}$$

for every nonnegative functional Φ. By (3.3), $H = P(W) + G - Q\log|\cdot|$ is the usual γ-inserted field up to the deterministic constant $-ac_K$.

7.2 Critical rooting

At $a = 0$, the ordinary zero-drift exponential integral diverges. A positive barrier truncation of derivative chaos instead has a Bessel root. The following calculation identifies the stationary law exactly.

Lemma 7.1 (Critical path measure)**.** *Let Q_L be the law of $X_t = L - R_t$, where R is a three-dimensional Bessel process started at $L > 0$. Put $M = \sup_{t\ge 0} X_t$ and $J = \int_0^\infty e^{\xi X_t}\,dt$. There is a sigma-finite measure N on these paths with*

$$\mathsf{N}|_{\{M<L\}} = L\mathsf{Q}_L, \qquad \mathsf{N}(J \in dj) = \xi^{-1} j^{-1} e^{-2/(\xi^2 j)}\,dj, \tag{7.2}$$
$$\mathsf{N}(J^{-1}) = \xi/2, \qquad d\mathbb{P}_*^- = (2/\xi) J^{-1}\,d\mathsf{N}.$$

The last probability law is the inward radial law of Proposition 3.2 at $a = 0$. In particular,

$$\frac{d\mathsf{Q}_L}{d\mathbb{P}_*^-} = \frac{\xi J}{2L}\mathbf{1}_{\{M<L\}}, \qquad L\mathbb{E}_{\mathsf{Q}_L}[\mathbf{1}_{\{J\le T\}}\Phi] \le \frac{\xi T}{2}\mathbb{E}_*\Phi\,. \tag{7.3}$$

Proof. For $L' > L$, a Bessel process started at L' avoids $L' - L$ forever with probability L/L'. Conditional on this event, the path $L' - R$ has law Q_L, by the Brownian killed-process Doob transform. Hence $L'\mathsf{Q}_{L'}|_{\{M<L\}} = L\mathsf{Q}_L$, defining N. Bessel transience and its Green kernel give $J < \infty$ on every such finite restriction.

For $z > 0$, removal of the Doob transform from the Feynman–Kac equation shows that

$$h_L(w) = (L - w)\mathbb{E}_w \exp\left(-z\int_0^\infty e^{\xi X_t}\,dt\right)$$

solves $h_L''/2 = ze^{\xi w}h_L$, with $h_L(L) = 0$ and $h_L(w) = -w + O(1)$ as $w \to -\infty$. Substituting $ce^{\xi w/2}$, where $c = 2\sqrt{2z}/\xi$, gives

$$h_L(w) = \frac{2}{\xi}\left[K_0(ce^{\xi w/2}) - \frac{K_0(ce^{\xi L/2})}{I_0(ce^{\xi L/2})} I_0(ce^{\xi w/2})\right].$$

Here I_0, K_0 are the two solutions of $r^2 f'' + rf' - r^2 f = 0$, normalized by $I_0(0) = 1$, while $K_0(r) = -\log r + O(1)$ at zero and K_0 decays at infinity. The coefficient and boundary conditions determine the displayed solution. Letting $L \to \infty$ at $w = 0$ gives $\mathsf{N}(e^{-zJ}) = (2/\xi)K_0(2\sqrt{2z}/\xi)$. The integral representation of this solution, obtained by substituting into its differential equation, is

$$\frac{2}{\xi}K_0(2\sqrt{2z}/\xi) = \frac{1}{\xi}\int_0^\infty e^{-zj - 2/(\xi^2 j)}\frac{dj}{j}\,.$$

Laplace-transform uniqueness proves the marginal in (7.2), and integration gives $\mathsf{N}(J^{-1}) = \xi/2$.

It remains to identify paths, not only J. Let $A_T = \int_0^T e^{\xi X_t}\,dt$. The Markov property of the killed Brownian transform, followed by $L \to \infty$, gives the finite-prefix density of the biased law relative to Brownian motion as

$$\int_0^\infty \frac{e^{-g}}{e^{\xi X_T} + (\xi^2/2)A_T g}\,dg\,.$$

Indeed the unnormalized future contribution is $\int \mathsf{N}(J \in dj)/(A_T + e^{\xi X_T} j)$; insert the marginal and multiply by $2/\xi$. Conditional on $J = 1/q$, this prefix density is

$$\frac{1}{1 - qA_T}\exp\left(\frac{2q}{\xi^2} - \frac{2qe^{\xi X_T}}{\xi^2(1 - qA_T)}\right)\mathbf{1}_{\{qA_T<1\}}\,.$$

Its endpoint logarithmic derivative is $-(2q/\xi)e^{\xi X_t}/(1 - qA_t)$. The resulting diffusion has the explicit solution

$$X_t = B_t - \frac{2}{\xi}\log\left(1 + q\int_0^t e^{\xi B_v}\,dv\right). \tag{7.4}$$

Its exponential integral through time t equals $A_t^B/(1+qA_t^B) < 1/q$, so it is global. Under the bias, q has exponential density $(2/\xi^2)e^{-2q/\xi^2}$. Reversibility of the stationary CIR diffusion in Proposition 3.2, followed by the physical-time change, gives exactly (7.4) for its negative physical times, while its positive physical times are independent Brownian motion conditional on q. This proves the identification and (7.3). □

7.3 Spatial summability

The subcritical rooting identity for the star-scale field is

$$\mathbb{E}\int \Phi(x,X)\,\mu_X(dx) = \int \mathbb{E}_0\Phi(x,X^{x,\gamma})\,dx\,.$$

For critical chaos, let Z^L denote the positive barrier truncation. Its identity is

$$\mathbb{E}\int \Phi(x,X)\,Z^L(dx) = L\int \mathbb{E}_{\mathsf{Q}_L}\Phi(x,X^{x,L})\,dx\,. \tag{7.5}$$

These formulas hold with additional noise or other functionals included. At a cutoff they are Gaussian change of measure, and, in the second case, its positive harmonic transform by $L+\gamma_c t - H_t(x)$ before hitting zero. Passage to the chaos limits gives the stated identities. The orthogonal Gaussian residual is unchanged. They are the rooting components of the background chaos results stated in Section 6.

Proposition 7.2 (Unrooted summability)**.** *Choose the parameters in Proposition 5.1 with the fixed perturbation bound $L_f = 1$. Set $r_j = \alpha^j$ and define $v_j(x)$ by (5.1), centered at x, for the ordinary metric D_h. For $s < q' < q$, almost surely, for every compact K and integer $m \geq 1$,*

$$\sum_{N=m}^{\infty} \alpha^{-q'N}\int_K \left(\prod_{j=m}^{N} v_j(x)\right)^p \mu_h(dx) < \infty\,. \tag{7.6}$$

Proof. Write $h = X + g$, with g locally smooth. Dependence between g and X is allowed, since $\mu_h = e^{\gamma g}\mu_X$ and the whole functional is rooted. Add one independent outer white noise jointly for every root. Its anchored outer field is

$$O_x(z) = \int_{r\geq 1}\int_{\mathbb{R}^d} r^{-(d+1)/2}\left[K\left(\frac{x+z-y}{r}\right) - K\left(\frac{x-y}{r}\right)\right]\mathcal{W}'(dy,dr)\,.$$

It is jointly smooth on compact root-coordinate sets. Put $C_*(z) = \int_0^\infty k(e^t z)\,dt$. The function $C_*(z) + \log|z|$ has a smooth extension through zero, as follows from (3.3). The actual rooted field near x equals the canonical field H plus the correction

$$f_x(z) = \gamma(C_*(z) + \log|z|) + ac_K + g(x+z) - O_x(z)\,. \tag{7.7}$$

This identity follows by separating fine and outer radial projections. The insertion shifts only fine noise; its fine covariance is C_*, whereas H has the full logarithmic insertion. At criticality $a = 0$.

Let $c_x = f_x(0)$. Joint continuity implies that on every compact set of roots, almost surely some integer N_0 satisfies

$$\sup_{x\in K,\ |z|\leq 3/N_0} |f_x(z) - c_x| \leq 1\,. \tag{7.8}$$

For fixed N_0, put $\rho = 1/N_0$. Multiply the centered correction by a cutoff equal to one on $B^E_{2\rho}$ and zero outside $B^E_{3\rho}$. Clip it first to $[-1,1]$, so its norm is at most one on all realizations. On (7.8) this changes nothing locally. The constant c_x is handled by metric rescaling.

By (6.12), J_X, its reciprocal, and $|c_x|$ are bounded uniformly over compact root sets. The escape distance $D_h(x, \partial B^E_\rho(x))$ is positive and continuous, hence has a positive compact minimum. Localize these quantities, and $|g(x)|$, by an integer T. The radius index N_0 and bound index T are independent choices. Under the rooted identity, the same functional J_X becomes precisely the canonical inward integral J. Thus $J \in [T^{-1}, T]$ on the rooted localization.

At criticality, Lemma 6.1 also gives $\sup_{x\in K, t\geq 0}(H_t(x) - Qt) < \infty$ almost surely. For a sufficiently large integer L the barrier is never hit on K. On that event $Z^L = \mu_X$ there: their cutoff densities differ by L times ordinary critical chaos, whose limit is zero. Include this barrier event as another localization. It may be discarded after applying (7.5). Equations (7.1) and (7.3) now bound both rooted expectations by a finite constant times $\mathbb{E}_*$. No first moment of the untruncated critical measure is used.

All sufficiently fine hulls, their boundary diameters, and relevant short paths are confined to $B^E_\rho(x)$ at a deterministic initial level depending on N_0, T. A path leaving that ball pays the localized escape cost. The root metric agrees there with the completed metric described in Section 5.1. Since $D_{H+f_x} = e^{\xi c_x} J D_{F+\widetilde{f}_x}$ locally, the actual initial metric scale is $\Lambda_x = e^{-\xi c_x}/J$, in a deterministic compact subinterval of $(0, \infty)$.

A uniform estimate for deterministic initial scales cannot be applied directly to this random scale. Divide the compact interval into finitely many bins $[\lambda_i, b\lambda_i]$, with one fixed $1 < b < 4/3$. Monotonicity bounds the radius ratio at any scale in the bin by the ratio with numerator radius $64b\lambda_i r_j$ and denominator radius $\lambda_i u_j/2$. Failure of F_* at the actual scale forces its diameter inequality on the deterministic interval

$$[3b\lambda_i r_{j-1}/16, \lambda_i r_{j-1}/4]$$

with threshold $32b\lambda_i u_j$. This interval is nonempty. These are a fixed set of multiplier changes, independent of the localization indices; λ_i is simply a deterministic R_0 in Proposition 5.1.

Apply that proposition to the finitely many bin majorants, after dropping the deterministic initial prefix. The rooted density comparison yields, for each fixed localization,

$$\mathbb{E}\int_K \mathbf{1}_{\text{localization}} \left(\prod_{j=1}^{N} v_j(x)\right)^p \mu_h(dx) \leq C_{K,L,N_0,T}\alpha^{qN}.$$

Multiply by $\alpha^{-q'N}$ and sum. Tonelli gives almost sure summability for each localization. Almost surely some indices cover every root in K simultaneously, so a countable union removes the indicators. Since $v_j \geq \eta$, deleting any fixed prefix costs only a finite factor. A countable compact exhaustion proves (7.6). □

8 One global quasisymmetric metric

We now work deterministically. The admissibility argument uses the Euclidean embedding to assign weights, but does not use a doubling property. The graph includes every integer scale, so its boundary metric is defined on all of $\mathbb{R}^d$ at once.

Put $r_j = \alpha^j$, $u_j = \alpha^{j-1+\zeta}$, and define

$$\sigma(x,j) = \left(\frac{\operatorname{diam}_E B^D_{4r_j}(x)}{\operatorname{in}_x H^\circ_{u_j}(x)} \wedge 1 \right) \mathbf{1}_{\widetilde{F}(x,j)} + \mathbf{1}_{\widetilde{F}(x,j)^c} \,. \tag{8.1}$$

Lemma 8.1 (Admissibility)**.** *Suppose consecutive balls $C_i = B^D_{4r_j}(x_i)$ overlap, the first meets $B^D_{r_{j-1}}(y)$, and the last meets $\mathbb{R}^d \setminus B^D_{2r_{j-1}}(y)$. Then*

$$\sum_i \sigma(x_i, j) \geq 1 \,. \tag{8.2}$$

Proof. Write $R = r_{j-1}$, $r = r_j$, $u = u_j$. If a weight is one, there is nothing to prove. Otherwise it suffices to find a consecutive subchain whose union C is not contained in $H^\circ_u(x_i)$ for any retained center. Then $\operatorname{in}_{x_i} H^\circ_u(x_i) \leq \operatorname{diam}_E C \leq \sum \operatorname{diam}_E C_i$, and summing the ratios gives (8.2).

Choose z in the final ball with $D(y,z) \geq 2R$. If z is in the unbounded component outside $B^D_{3R/2}(y)$, use the nested hulls $K_t = H_{R+t}(y)$ and inner point $a_0 = y$. Otherwise reverse the chain and let K_t be the component containing z of $\mathbb{R}^d \setminus B^D_{2R-t}(y)$, with $a_0 = z$. Use both open and closed conventions. In either case the chain crosses from $K_0^\bullet$ to the complement of $K^\circ_{R/2}$, the boundaries and complementary continua are connected, and paths between levels $v < w$ pay at least $w - v$. These statements follow from Lemma 2.1; in the second case every exit from the selected component crosses the indicated root-distance level.

Trim at first exit and last entry across the corridor between $K^\bullet_{2u}$ and the complement of K°_{4u}. Every retained ball meets this closed corridor. If $a_0 \notin H^\circ_u(x_i)$ for every retained center, its u-ball misses at least one of the corridor's two connected sides, which are $2u$ apart. The missed outer side belongs to the unbounded component of the ball complement. If the inner side is missed instead, it contains a_0 and hence belongs to that same unbounded component. A corresponding endpoint of the subchain is therefore outside $H^\circ_u(x_i)$, as required.

It remains to consider a center x_k that swallows a_0. The distance from x_k to K_0 is at least $2u - 4r > u$, so the connected set K_0 lies in the same bounded complementary component as a_0. Thus $K_0^\bullet \subset H^\circ_{R/16}(x_k)$. The exterior of $K^\circ_{R/2}$ is connected and unbounded, and its distance from x_k is at least $R/2 - 4u - 4r > 5R/16$. Hence

$$K_0^\bullet \subset H^\circ_{R/16}(x_k), \qquad H^\circ_{5R/16}(x_k) \subset K^\circ_{R/2} \,.$$

The witness for $\widetilde{F}(x_k, j)$ supplies a good band whose inner hull contains K_0 and whose outer hull is inside $K_{R/2}$. The chain crosses this band.

Use its intrinsic arrival coordinate and trim across the layers $3u$ and $5u$. Their ambient distance is at least $2u$: paths staying in the band pay by Lemma 2.2, while a path leaving it first pays at least $3u$. Since $4r < u/16$, every retained center is in the intrinsic middle layer $[2u, 6u]$, and its ambient u-ball equals its internal ball. Such a ball misses one of the two sides. The missed outer side connects to the exterior without meeting the ball. The missed inner side connects to the inner boundary, which goodness places in the exterior component of the ball complement. In either case the corresponding endpoint of the subchain is outside the filled u-ball. This proves the required property of C. □

For $k \geq 0$ define $\tau_k(x,j) = (\eta + 2 \sup_{D(x,y) \leq k r_j} \sigma(y,j)) \wedge 1$. These functions need not be measurably selected; they are used only in deterministic inequalities.

Lemma 8.2 (Local comparison)**.** *For sufficiently small α,*

$$\tau_{27}(x,j) \le v_j(x)\,. \tag{8.3}$$

If $D(x,z) \le r_N/3$ and $j \le N$, then $\tau_{26}(x,j) \le \tau_{27}(z,j)$.

Proof. The second assertion is the triangle inequality. If $F_*(x,j)$ fails, $v_j(x) = 1$. Otherwise (2.2) gives $\widetilde{F}(y,j)$ for $D(x,y) \le 27r_j$. Put $U = \operatorname{out}_x H^{\bullet}_{64r_j}(x)$ and $V = \operatorname{in}_x H^{\circ}_{u_j/2}(x)$. Hull inclusion gives $B^D_{4r_j}(y) \subset H^{\bullet}_{64r_j}(x)$ and $H^{\circ}_{u_j/2}(x) \subset H^{\circ}_{u_j}(y)$, provided $27r_j \le u_j/2$. Also $|x-y| \le U$. If $V > 2U$, the ratio in (8.1) is at most $2U/(V-U) \le 4U/V$. For $V \le 2U$ the same upper bound follows from $\sigma \le 1$. Thus $\tau_{27} \le (\eta + 8U/V) \wedge 1 \le v_j$. □

8.1 Weights and edges

For each $j \in \mathbb{Z}$, choose a maximal r_j-separated net A_j. Properness makes it countable and locally finite. Vertices (x,j) and (y,j) are horizontal neighbors when their open $4r_j$-balls overlap. Include self-neighbors in extrema. Choose a parent $p(v)$ in level $j-1$ at distance at most r_{j-1} from v.

Put $\beta = 32\alpha < 1/2$ and choose $\eta < \beta$. For $j \le 0$, set $\pi(v) = \beta^j$. At $j \ge 1$, define

$$\begin{aligned} \nu(v) &= 2\max\{\sigma(w,j) : v \sim z \sim w\}, & \mu(v) &= \max\{\eta, \nu(v)\} \wedge (1-\eta), \\ \pi_0(v) &= \pi(p(v))\mu(v), & \pi(v) &= \max\{\pi_0(v), \eta \max_{w\sim v} \pi_0(w)\}. \end{aligned} \tag{8.4}$$

Every maximum is over a finite set. Write $\rho(v) = \pi(v)/\pi(p(v))$. Then

$$\eta \le \rho(v) \le 1-\eta, \qquad \eta \le \frac{\pi(v)}{\pi(w)} \le \eta^{-1} \ (v \sim w)\,; \qquad \rho(v) = \beta \ (j \le 0)\,. \tag{8.5}$$

Here and below α is small enough that parents of vertices within three horizontal steps are neighbors. To verify the induction, the extra smoothing term is at most $\eta\eta^{-1}(1-\eta)\pi(p(v))$. For horizontal comparison, the term $\pi_0(v)$ is at most $\eta^{-1}\pi(w)$. If $z \sim v$, then $p(z) \sim p(w)$, and $\eta\pi_0(z) \le (1-\eta)\pi(p(w)) \le \eta^{-1}\pi(w)$. These two estimates prove (8.5).

For a horizontal chain $v_0, \dots, v_N$ crossing an annulus of radii r_{j-1} and $2r_{j-1}$,

$$\sum_{i=1}^{N} \min_{w\sim v_{i-1} \text{ or } w\sim v_i} \rho(w) \ge 1\,. \tag{8.6}$$

For $j \ge 1$ each summand is at least η and at least

$$\min\{1-\eta,\, 2\max(\sigma(v_{i-1},j), \sigma(v_i,j))\}\,.$$

If one of these latter bounds reaches $1-\eta$, another edge supplies η; a crossing has at least two edges. Otherwise summing bounds the sum of the vertex σ's, which is at least one by Lemma 8.1. For $j \le 0$, a crossing has at least $(16\alpha)^{-1}$ edges, each with ratio $\beta = 32\alpha$.

Give each parent or horizontal edge (v,w) length $\pi(v) + \pi(w)$, and let d_G be the graph path metric. The following full finite-path argument supplies its lower bound.

Lemma 8.3 (Finite-path comparison)**.** *For distinct vertices v, w, let k be the largest level, no greater than either endpoint level, at which their level-k ancestors are neighbors. Set $P(v,w)$ equal to the maximum of the two ancestor weights there. Then*

$$cP(v,w) \le d_G(v,w) \le CP(v,w)\,. \tag{8.7}$$

The constants depend only on α, η.

Proof. Such a level exists because $r_j \to \infty$ as $j \to -\infty$ and an ancestor displacement is at most $r_j/(1-\alpha)$. Following ancestors to level k and joining horizontally proves the upper bound by geometric summation in (8.5).

For the lower bound put

$$\pi_*(v) = \min_{z \sim v} \pi(z)\,.$$

Give a horizontal edge auxiliary length $\min(\pi_*(v), \pi_*(w))$ and a parent edge $(p(v), v)$ auxiliary length $\eta^{-4}\pi_*(v)$. These lengths are at most a fixed multiple of the original lengths. Every incident edge at v has auxiliary length at least $\eta^2\pi(v)$.

Step 1: a crossing pays a parent weight. Trim a level-j horizontal chain crossing $(R, 2R)$ about a level-$j-1$ vertex z at its first exit. Every retained ball meets $B^D_{2R}(z)$. The parent of any neighbor of a retained vertex is a neighbor of z, since its center is at distance at most $3R + 12r_j < 8R$. Therefore $\pi(w) \geq \pi_*(z)\rho(w)$. Equation (8.6) shows that the auxiliary cost of the crossing is at least $\pi_*(z)$.

Step 2: remove a one-level excursion. Suppose a, b lie at level n and $a, w_1, \ldots, w_m, b$ is a path with all w_i at level $n+1$. There is a horizontal level-n replacement whose cost is no greater than the cost of $w_1, \ldots, w_m, b$; the first edge (a, w_1) is not charged. If a, b are neighbors, the last parent edge pays their horizontal edge, since $\eta^{-4}\pi_*(w_m) \geq \pi(b)$. Otherwise stop the fine chain at its first crossing of $(R, 2R)$ about a. By Step 1 this pays $\pi_*(a)$ and hence the edge from a to the parent a_1 of the crossing endpoint. Restart from that endpoint with center a_1. The crossing index increases strictly: a fine point is within R of its parent. The completed crossing index intervals are disjoint. Continue until the current parent neighbors b; the last original parent edge pays the remaining coarse edge. If the remaining tail makes no further crossing, that neighboring relation follows from its containment in B_{2R} and the distance at most R to b.

Step 3: project a finer endpoint. Excursions above a prescribed endpoint level can be removed from the maximum level downward by Step 2. If the endpoint levels differ, reverse the path and take the initial segment from the finer endpoint to its first visit one level lower. Prepend the fictitious parent edge of that endpoint. Step 2 removes this excursion without charging the fictitious edge. Repeat until the endpoint levels agree. All these operations decrease auxiliary length.

Step 4: project both endpoints. Suppose the endpoints have the same level n and their parents are not neighbors. Remove all excursions above n. If the path visits level $n-1$, apply Step 2 to the two endpoint tails using fictitious first edges, then remove the remaining fine excursions. If the path is entirely horizontal, partition it greedily into completed crossings about the parent of the current starting point. Each crossing pays that parent's weight. When the tail no longer crosses, omit the last intermediate parent and join the preceding parent directly to the target parent. This last edge is legal: its centers are at distance at most $6R + 4r_n < 8R$. Its cost is paid by the last completed crossing. At least one crossing occurs because the endpoint parents are not neighbors. Thus both endpoints are projected one level down at no greater cost.

Step 5: stop at the separating level. If k is already an endpoint level, an incident edge costs at least a fixed multiple of $P(v, w)$, using horizontal comparison. Otherwise use Steps 3–4 until both endpoints are at level $k+1$. They are distinct, while their parents at level k are neighbors. The reduced path is nonempty. Its first and last edges have cost at least η^2 times the corresponding child weight, each at least η times its parent weight. Thus its cost is at least $\eta^3 P(v, w)$. All replacements decrease cost. Comparison of auxiliary and original lengths proves the lower bound. Each procedure operates on a finite path and terminates at a finite level; no lowest integer scale or bounded degree is needed. □

Proposition 8.4 (Boundary metric)**.** *Fine net approximations to $x \in \mathbb{R}^d$ define one completion point $\iota(x)$ for d_G. The pullback $\delta(x,y) = d_G(\iota(x),\iota(y))$ is a metric globally quasisymmetric to D. Moreover, for $x \in A_N$, $N \geq 1$,*

$$\operatorname{diam}_\delta B^D_{r_N}(x) \leq C\pi(x,N), \qquad \pi(x,N) \leq (1-\eta)^N . \tag{8.8}$$

For $D(x,z) \leq r_N/3$ and every integer $1 \leq m \leq N$,

$$\pi(x,N)^p \leq \left(\prod_{j=m}^{N} v_j(z) \right)^p . \tag{8.9}$$

Proof. Choose $x_j \in A_j$ with $D(x,x_j) < r_j$. Consecutive choices are connected by a parent edge and a horizontal edge. Their weights are $O((1-\eta)^j)$, so they are Cauchy; different choices have vanishing distance.

For $x \neq y$, choose k with $r_{k+1} < D(x,y) \leq r_k$. At level k, the ancestors of sufficiently fine approximations are neighbors; at level $k+2$ they are not, since their displacement from the limiting points is $O(r_{k+2}) \ll r_{k+1}$. Lemma 8.3 therefore gives

$$\delta(x,y) \asymp \pi(z_k(x),k), \tag{8.10}$$

where $z_k(x)$ is any level-k net point within r_k of x. This proves injectivity. For triples, horizontal comparison and the vertical bounds give

$$\frac{\delta(x,y)}{\delta(x,z)} \leq \begin{cases} C(D(x,y)/D(x,z))^{\log(1-\eta)/\log\alpha}, & D(x,y) \leq D(x,z), \\ C(D(x,y)/D(x,z))^{\log\eta/\log\alpha}, & D(x,y) > D(x,z). \end{cases}$$

Both powers are positive. This continuous piecewise power bound is an admissible distortion function after increasing C. It is global, including the coarse levels. Formula (8.10) also gives (8.8).

A level-j ancestor of (x,N) is within $2r_j$ of x. Dividing (8.4) by the parent weight, horizontal comparison shows that the ratio is at most the largest μ among one-step neighbors. Its σ terms are at most three steps away, hence their centers are within $26r_j$ of x. Therefore $\pi(x,N) \leq \prod_{j=1}^N \tau_{26}(x,j)$. Apply Lemma 8.2, and drop the initial factors, each at most one, to get (8.9). □

9 The dimension calculation

Proof of Theorem 1.2. Fix $p > d$ and choose the parameters in Proposition 7.2. Construct δ_p once on the whole space by Proposition 8.4. Let K be compact and V a slightly larger compact neighborhood. For large N, the net points whose r_N-balls meet K lie in V; their $r_N/3$-balls are disjoint and lie in a fixed enlargement of V. Choose $s < q_0 < q' < q$. Proposition 6.2 and (8.9) give

$$\begin{aligned} \sum_{x:\, B^D_{r_N}(x) \cap K \neq \varnothing} \pi(x,N)^p &\leq C\alpha^{-q_0 N} \sum_x \int_{B^D_{r_N/3}(x)} \left(\prod_{j=m}^{N} v_j(z) \right)^p \mu_h(dz) \\ &\leq C\alpha^{-q_0 N} \int_V \left(\prod_{j=m}^{N} v_j(z) \right)^p \mu_h(dz) \longrightarrow 0 . \end{aligned}$$

Rename the fixed enlargement V in the last line. The limit follows from (7.6), since $q' > q_0$. The covering diameters tend uniformly to zero by (8.8). Hence $\mathcal{H}^p_{\delta_p}(K) = 0$. A countable compact exhaustion gives the same conclusion on all of $\mathbb{R}^d$ for this one target metric.

For the lower bound, restrict any compatible target metric δ to a compact topological cube C. Its opposite faces F_i^-, F_i^+ have positive δ-distance. The functions

$$f_i(x) = \frac{\delta(x, F_i^-)}{\delta(x, F_i^-) + \delta(x, F_i^+)}$$

are Lipschitz, since their denominators are bounded below, and equal 0 and 1 on the respective faces. The Poincaré–Miranda theorem makes $f = (f_1, \dots, f_d)$ onto $[0,1]^d$. To see this from Brouwer, for $y \in (0,1)^d$ take a fixed point of the coordinatewise projection of $x + \varepsilon(y - f(x))$ onto the cube. Its boundary-face signs force it to be interior in every coordinate, and then $f(x) = y$. Compactness supplies boundary values. Lipschitz maps do not increase Hausdorff dimension, so $\dim_{\mathrm{H}}(C, \delta) \geq d$. Every quasisymmetric target has the required topology. Finally intersect the probability-one events for $p = d + 1/n$ to obtain (1.4). □

9.1 The quantum sphere

Proof of Corollary 1.4. The planar assertion follows from Theorem 1.2 and the canonical metric inputs cited in Section 1.

For the sphere fix $0 < \gamma < 2$ and three marked points. In the three-insertion representation let h^L be a spherical GFF plus the three γ-logarithmic insertions and its smooth background term. Write $Z = \mu_{h^L}(S^2)$ and $b = 4/\gamma^2 - 2$. Under the probability law

$$\frac{Z^b}{\mathbb{E}Z^b}\, d\mathbb{P},$$

the field $h^L + \gamma^{-1}\log(A/Z)$ represents the sphere of area $A > 0$ [1, Definition 2.2 and Theorem 2.3]. The density is positive and finite. For $b > 0$ its required moment is below the insertion threshold $4/\gamma^2 - 1$; for $b \leq 0$ negative chaos moments suffice. Thus probability-one statements for h^L survive this tilt. We construct the target metric before the random area normalization.

Give S^2 a round auxiliary metric d_0 of diameter one. By uniform compatibility of its topology with the compact LQG metric, there is $r_0 > 0$ such that every D-ball of radius $100r_0$ has round diameter less than $1/100$. Its complement has a unique component with round diameter greater than $1/2$: the component containing the complement of a small round cap that contains the ball. Define the exterior to be this component. All other components lie in the cap. This convention is independent of charts and agrees with planar filling inside any sufficiently small coordinate disk containing the ball.

Choose global nets on the sphere. Start at an integer level whose radius exceeds its D-diameter, with one coarse vertex. Use constant ratios β at the finitely many coarse levels, and the weights (8.1)–(8.4) once $r_{j-1} < r_0$. Replace Euclidean diameters and inradii by their round versions. The proof of Lemma 8.1 only uses the triangle inequality for this auxiliary metric. A chain can be trimmed at its first exit from $B^D_{2r_{j-1}}(y)$, so the retained balls lie in $B^D_{3r_{j-1}}(y)$ and in one chart. The local planar admissibility proof therefore applies with a point outside the chart in place of infinity. The finite-path proof gives one globally quasisymmetric metric δ_p on the sphere. Finite coarse factors only change the constants in (8.8) and (8.9).

On every compact coordinate disk avoiding the three marks, h^L is locally absolutely continuous with respect to a whole-plane GFF plus a smooth function. Locality transfers Propositions 6.2

and 7.2. A fixed finite round atlas has uniformly bounded comparisons between Euclidean and round distances. Those fixed constants are included when choosing the multiplier variants in Proposition 5.1. A smooth correction is centered and localized as in (7.8), so the same parameters work on the whole sphere.

Fix a compact K avoiding the marked points and a slightly larger neighborhood V that also avoids them. At sufficiently fine scales, its hulls, short paths, and observables are local to V. Apply the covering argument above to global net balls meeting K, dropping a finite initial prefix. It gives $\mathcal{H}^p_{\delta_p}(K) = 0$. Exhaust the complement of the three marks by countably many such compact sets and then add the three points. Thus $\mathcal{H}^p_{\delta_p}(S^2) = 0$. The topological-cube lower bound gives dimension at least two. Take $p = 2 + 1/n$.

Finally, fixing area multiplies the metric by $(A/Z)^{\xi/\gamma}$, a positive scalar. This leaves conformal dimension unchanged and proves the assertion for every fixed $A > 0$. □

10 Subcritical weak metrics and simultaneous covariance

We now remove exact dilation covariance from the subcritical application. A weak subcritical exponential metric has locality, the length-space and topology properties, simultaneous continuous Weyl scaling, fixed-translation covariance, and tightness of the normalized metrics across scales. The deterministic normalizations in that definition may be chosen as $r^{\xi Q}$; this follows from [2, Theorem 1.6]. We use the across and connection moments in Proposition 1.1(iii), together with one positive moment of the internal diameter of an annular sphere. More precisely, for some $b > 0$,

$$\sup_{z,r} \mathbb{E}\left[\left(r^{-\xi Q} e^{-\xi h_r(z)} \operatorname{diam}_{D_h}(\partial B^E_r(z); B^E_{2r}(z) \setminus \overline{B}^E_{r/2}(z))\right)^b\right] < \infty\,. \tag{10.1}$$

A fixed change of the buffer radii is harmless. These are established weak-metric inputs in the subcritical regime [2, Propositions 4.7, 4.10 and 4.11]. In this section they replace the dilation rule; the weak theory supplies them without that rule. We prove the remaining reduction.

Proposition 10.1 (Uniform comparison of rescalings)**.** *For $\ell > 0$ put*

$$D^\ell_h(x, y) = \ell^{-\xi Q} D_{h(\cdot/\ell)}(\ell x, \ell y)\,. \tag{10.2}$$

There is a deterministic $C < \infty$, independent of ℓ, such that for every fixed $\ell > 0$, almost surely

$$C^{-1} D_h \le D^\ell_h \le C D_h \tag{10.3}$$

simultaneously on all pairs of points. The same inequalities hold for internal metrics.

Proof. Pin h by $h_1(0) = 0$ and put $H = h(\cdot/\ell) - h_{1/\ell}(0)$. The field H has the same pinned law as h. Constant Weyl scaling gives the pathwise identity

$$r^{-\xi Q} e^{-\xi h_r(0)} D^\ell_h(rx, ry) = (r\ell)^{-\xi Q} e^{-\xi H_{r\ell}(0)} D_H(r\ell x, r\ell y)\,. \tag{10.4}$$

Thus the normalized crossing and annular-diameter tails are uniform in r and ℓ. Translation gives the same uniformity in the center.

We need a Gaussian product estimate for centered restrictions of the full field. The proof of Lemma 3.1 applies here as well. To check its two inputs, represent a linear observable

of $h(z + r\cdot) - h_r(z)$ by a zero-mass distribution supported in a fixed annulus, including the averaging sphere. Its covariance norm is a positive constant times

$$\int_{\mathbb{R}^d} |\widehat{\nu}(\omega)|^2 |\omega|^{-d}\, d\omega \geq c\|\nu\|^2_{H^{-d/2}}\,.$$

For two separated rescaled annuli, zero total mass permits replacing the cross kernel by $-\log|x - \rho y| + \log|x|$, where $\rho = e^{-H}$. All of its derivatives on the fixed annuli are $O(\rho)$. Sobolev duality therefore gives exponentially decaying maximal correlation. The covariance-domination proof of (3.4) gives the product estimate, also for different functionals at different scales.

Fix ℓ and write $D_0 = D_h$, $D_1 = D_h^\ell$. Let $E(z, r; A)$ require, for both $i = 0, 1$,

$$D_i(\text{across } B_r^E(z) \setminus \overline{B}^E_{r/2}(z)) \geq A^{-1} r^{\xi Q} e^{\xi h_r(z)},$$
$$\operatorname{diam}_{D_i}(\partial B_r^E(z); B_{2r}^E(z) \setminus \overline{B}^E_{r/2}(z)) \leq A r^{\xi Q} e^{\xi h_r(z)}.$$

By (10.4), the probability of failure tends to zero as $A \to \infty$, uniformly in z, r, ℓ. Locality and constant Weyl scaling make the event measurable from a centered field restriction in a fixed logarithmic shell window. Choose a sufficiently separated fixed fraction of the dyadic radii in

$$[\varepsilon^{3/4}, \varepsilon^{1/2}/10].$$

There are at least $c\log(1/\varepsilon)$ such radii. Applying (3.4) to their failure indicators, and then choosing A large, gives

$$\mathbb{P}[\text{none of these radii is good at } z] \leq \varepsilon^{2d+4}\,.$$

There are $O(\varepsilon^{-2d})$ points of $\varepsilon\mathbb{Z}^d$ in $B^E_{\varepsilon^{-1}}$. A union bound and Borel–Cantelli show that, along dyadic $\varepsilon \downarrow 0$, every such point eventually has a good radius. The constant A is independent of ℓ.

Here is the path replacement on that event. Let $P : [0, T] \to U$ be a D_0-arclength path in a bounded open set. At $t_0 = 0$, choose a nearest grid point z_0, a good radius r_0 there, and let t_1 be the first exit from $B^E_{r_0}(z_0)$. Repeat at each exit, and stop when no further exit occurs. For small ε, every starting point is inside the half-radius ball. Uniform continuity of P and the positive lower bound $r_i \geq \varepsilon^{3/4}$ imply that there are finitely many completed steps. Each satisfies

$$\operatorname{diam}_{D_1}(\partial B^E_{r_i}(z_i); B^E_{2r_i}(z_i) \setminus \overline{B}^E_{r_i/2}(z_i)) \leq A^2(t_{i+1} - t_i)\,. \tag{10.5}$$

Maintain a stack of the completed Euclidean balls in chronological order. A new ball contains the preceding exit point, so it overlaps the previous ball and is not contained in it. If their spheres meet, append it. Otherwise the previous ball is contained in the new ball: discard the previous ball and repeat. After a discard the same two properties hold with the preceding stack ball, because the new ball contains the discarded one. Thus consecutive retained spheres meet. Every discarded ball lies in a later retained ball. The first retained ball contains $P(0)$, and the last contains the last exit on its sphere. The uncharged final part of P remains in one ball of radius at most $\varepsilon^{1/2}/10$.

Choose points on successive sphere intersections and connect them using the internal-diameter bounds. Choose an arbitrary point on the first retained sphere and the last exit on the last sphere as endpoints. Their Euclidean distances from $P(0)$ and $P(T)$ tend to zero. The sum of the connection costs is at most A^2T, up to an arbitrarily small approximation error, since the retained increments in (10.5) are disjoint. If no step is completed, the two endpoint neighborhoods intersect. Letting $\varepsilon \downarrow 0$ and using continuity of D_1 gives

$$D_1(P(0), P(T)) \leq A^2 T\,.$$

Take paths approaching the internal distance in U, and then exhaust $\mathbb{R}^d$ by bounded open sets. Every path image is compact, so the length-space property gives $D_1 \le A^2 D_0$. Interchanging D_0, D_1 proves the reverse bound. These comparisons imply the same bounds for lengths of every path, and hence for internal metrics. This proves (10.3) with $C = A^2$. No simultaneous exceptional-event assertion over ℓ was needed. □

Proposition 10.2 (A covariant representative)**.** *Suppose a measurable local length metric inducing the Euclidean topology, with fixed-translation covariance and simultaneous continuous Weyl scaling, satisfies* (10.3) *for every fixed ℓ, with one deterministic C. There is a measurable local length metric L_h, bi-Lipschitz equivalent to D_h, which satisfies continuous Weyl scaling, fixed-translation covariance, and simultaneous dilation and additive-constant covariance. All local annular moment bounds transfer to L_h. If $D_h^\ell = D_h$ almost surely for every fixed ℓ, the representative equals D_h almost surely.*

Proof. Take an essential supremum with respect to Haar measure $d\ell/\ell\, dc$:

$$q_h(x,y) = \operatorname*{ess\,sup}_{\ell>0,\ c\in\mathbb{R}} \ell^{-\xi Q} e^{-\xi c} D_{h(\cdot/\ell)+c}(\ell x, \ell y)\,. \tag{10.6}$$

One may use any strictly positive integrable density relative to this measure to define the same essential supremum. Joint measurability, Fubini, constant Weyl scaling, and (10.3) give the inequalities $C^{-1}D_h \le q_h \le CD_h$ first on a countable dense set of pairs, and then on all pairs by continuity. Indeed the family under the supremum is uniformly C-Lipschitz in each argument with respect to D_h. Its essential supremum is therefore a continuous metric.

Changing variables (ℓ, c) in (10.6) gives, algebraically and simultaneously for all $t > 0$, $b \in \mathbb{R}$,

$$q_{h(t\cdot)+b}(x,y) = t^{-\xi Q} e^{\xi b} q_h(tx, ty)\,. \tag{10.7}$$

Let L_h be the induced length metric of q_h. Since D_h is a length metric,

$$C^{-1}D_h \le q_h \le L_h \le CD_h\,.$$

Formula (10.7) passes to path lengths and L_h.

We verify locality, which does not follow merely from a supremum of global distances. For an open set U, define q_h^U by replacing every distance in (10.6) with its internal distance in ℓU. The resulting function is measurable from $h|_U$. To justify this assertion without choosing uncountably many local versions, test the essential supremum against a rational threshold. The event that it exceeds that threshold is equivalent to positivity of the integral of the corresponding indicator against a positive integrable density. Conditional Fubini makes this integral $h|_U$-measurable. Use a countable dense set of pairs.

If $K \Subset U$, its D_h-distance to U^c is positive. Uniform comparison implies that a sufficiently short path for almost every rescaled metric, uniformly in its index and starting in K, cannot exit U: an exit costs at least $C^{-1}\operatorname{dist}_{D_h}(K, U^c)$, whereas a short D_h-path has cost at most $CD_h(x,y)$. Consequently $q_h = q_h^U$ near the diagonal of $K \times K$. Every compact path in U has the same length for these two functions. Taking path infima proves locality of the internal metric of L_h.

For Weyl scaling, simultaneous Weyl scaling of D and Fubini give, for almost every (ℓ, c), the Weyl rule for the rescaled metric simultaneously for every continuous f. The comparison with D_h also passes to the weighted path lengths. On a small neighborhood where $\operatorname{osc} f \le \varepsilon$, the same positive escape-cost argument confines all relevant short paths, uniformly in (ℓ, c). Thus

the corresponding short distances before and after adding f are bounded by the factors $e^{\xi \inf f}$ and $e^{\xi \sup f}$. Taking essential suprema and refining path partitions yields

$$\operatorname{len}_{q_{h+f}}(P) = \int_P e^{\xi f}\, d \operatorname{len}_{q_h}$$

for every rectifiable compact path. A metric and its induced length metric give every path the same length. Taking infima over paths proves $L_{h+f} = e^{\xi f} \cdot L_h$. This argument is local; f need not be bounded on $\mathbb{R}^d$.

The construction is measurable. One explicit verification uses shortest small-mesh chains with vertices in a fixed countable dense subset of each compact neighborhood. As the mesh tends to zero, bounded chain lengths give continuous limits on that compact set, by the elementary compactness proof for length spaces. Thus the limit equals the infimum of lengths of paths in the compact neighborhood. Exhausting an open set by nested compact neighborhoods gives its internal length metric as a Borel function. This also handles a continuous perturbation that destroys global properness. Fixed-translation covariance follows from that of the rescaled metrics and Fubini. Uniform comparison transfers all local annular estimates.

Finally, if every fixed rescaling equals D_h almost surely, take $C = 1$. Then $q_h = L_h = D_h$ almost surely, while (10.7) still holds on the full orbit. □

Proof of Corollary 1.3. For every weak subcritical exponential metric, Proposition 10.1 supplies the uniform comparison and Proposition 10.2 supplies L_h satisfying Proposition 1.1. The proof of Theorem 1.2 uses only these properties and therefore gives $\operatorname{Cdim}(\mathbb{R}^d, L_h) = d$. The identity between D_h and L_h is bi-Lipschitz and hence quasisymmetric, so their conformal dimensions coincide. For the constructed strong metrics in every dimension, including the critical parameter, apply Proposition 10.2 directly with $C = 1$, using exact covariance at each fixed dilation. This provides the simultaneous version without changing the metric. Proposition 10.1 and its subcritical diameter estimate are not needed for this application. □

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