A higher-dimensional nonexistence theorem for local Q-covariant measures

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Abstract

For every $d \geq 2$ and $0 < Q < \sqrt{2d}$, we prove the zero-or-infinite-on-every-open-set conclusion of Bhatia–Gwynne–Sung for measures locally determined by a unit-log whole-space log-correlated Gaussian field. Only covariance under translations and dilations is needed. The proof uses an exact Gaussian scale decomposition, recurrence of a centered one-dimensional Gaussian random walk, and the Palm identity for auxiliary subcritical Gaussian multiplicative chaos with parameter Q. No Weyl-scaling axiom, local finiteness assumption, or moment assumption on the candidate measure is imposed. The proof explicitly handles the singular rooted field and the exceptional sets in deterministic-domain locality.

1 Statement and conventions

A unit-log whole-space log-correlated Gaussian field (LCGF), modulo additive constants, has covariance

$$\operatorname{Cov}\left((H,\varphi),(H,\psi)\right) = \iint_{\mathbb{R}^d\times\mathbb{R}^d} \log\frac{1}{|x-y|}\,\varphi(x)\psi(y)\,dx\,dy \tag{1}$$

for smooth compactly supported test functions of integral zero. This fixes the normalization of the logarithmic singularity; see [2]. To avoid an unspecified additive constant, use the actual representative H constructed in Section 2, or $\widehat{H} = H + C$, where C is an independent standard normal variable. These two choices have equivalent restrictions to every bounded open set, as proved below.

Definition 1.1. A random distribution h on a domain $U \subseteq \mathbb{R}^d$ is *admissible* if each point of U has a deterministic relatively compact open neighborhood $V \subset U$ on which

$$\operatorname{Law}(h|_V) \ll \operatorname{Law}(\widehat{H}|_V).$$

The neighborhood may depend on the law of h, but not on its realization.

The same admissible class is obtained from the usual whole-space LCGF pinned by a compactly supported smooth average; see Lemma 2.2 below.

The following statement is a higher-dimensional version of [1, Theorem 1.5]. The proof given here is based on Gaussian scale recurrence, rather than a loop ensemble.

Theorem 1.2. *Fix $d \geq 2$ and $0 < Q < \sqrt{2d}$. For each domain $U \subseteq \mathbb{R}^d$, suppose we have a measurable assignment*

$$\mathfrak{m}^U : \mathcal{D}'(U) \longrightarrow \{\textit{nonnegative Borel measures on } U\}.$$

The measure space is equipped with the sigma-algebra generated by the evaluation maps $\mu \mapsto \mu(A)$ for Borel A. The measures may take the value $+\infty$, and no sigma-finiteness is assumed. Suppose the following hold whenever h is admissible.

(i) ***Locality.*** *For every deterministic subdomain $V \subseteq U$, almost surely*

$$\mathfrak{m}_h^U|_V = \mathfrak{m}_{h|_V}^V. \tag{2}$$

(ii) ***Translation and dilation covariance.*** *For every deterministic map $f(x) = a + rx$, with $r > 0$ and $f : \widetilde{U} \to U$, almost surely*

$$\mathfrak{m}^{\widetilde{U}}_{h\circ f + Q\log r}(A) = \mathfrak{m}_h^U(f(A)) \qquad \textit{for every Borel } A \subseteq \widetilde{U}. \tag{3}$$

Then exactly one of the following alternatives holds, with the same alternative for all admissible fields and all domains:

(a) $\mathfrak{m}_h^U = 0$ *almost surely for every admissible h.*

(b) *Almost surely, $\mathfrak{m}_h^U(O) = \infty$ for every nonempty open $O \subseteq U$, for every admissible h.*

In particular, there is no nonzero, locally finite assignment satisfying these axioms.

For a conformal map in dimension d, the appropriate local scale is

$$\lambda_f(x) = |\det Df(x)|^{1/d}.$$

Thus the full conformal coordinate rule would use $h \circ f + Q \log \lambda_f = h \circ f + (Q/d) \log |\det Df|$. Such a rule implies (3), so the theorem applies. If the covariance has leading coefficient $\kappa \log(1/|x-y|)$, rescale the field by $\sqrt{\kappa}$: the threshold is then $Q < \sqrt{2d\kappa}$.

Remark 1.3. There is no assumption that adding a constant to h multiplies $\mathfrak{m}_h$ by an exponential, and no assumption that $\mathbb{E}\mathfrak{m}_h(A)$ is finite. Recurrence will control an arbitrary measurable dependence of $\mathfrak{m}_h$ on the additive constant.

2 An exact scale decomposition and local equivalence

Choose a nonnegative radial function $\phi \in C_c^\infty(B(0,1/2))$ with $\|\phi\|_{L^2} = 1$. Set $\check{\phi}(x) = \phi(-x)$ and

$$k = \phi * \check{\phi}.$$

Then k is smooth, radial, positive definite, supported in $\overline{B}(0,1)$, and $k(0) = 1$. Define

$$K(z) = \int_0^\infty k(e^t z)\, dt, \qquad z \neq 0. \tag{4}$$

Let X be the stationary centered Gaussian distribution with covariance $K(x-y)$.

Lemma 2.1. *The following statements hold.*

(i) $K(z) = \log(1/|z|) + g(z)$ *for* $z \neq 0$, *where* $g \in C^\infty(\mathbb{R}^d)$.

(ii) *Every compactly supported smooth function is a Cameron–Martin shift for* X. *Consequently, adding a deterministic function smooth on a neighborhood of a bounded set preserves the local equivalence class of its law.*

(iii) *There is a whole-space representative* $H = X + Y$, *where* Y *is an independent locally smooth Gaussian function. On every bounded open set, the laws of* X, H, *and* $\widehat{H} = H + C$ *are mutually absolutely continuous.*

(iv) *One can realize* H *from iid smooth stationary Gaussian fields* $(F_n)_{n\in\mathbb{Z}}$ *so that, for the left shift* $(\theta\omega)_n = F_{n+1}$,

$$H_\omega(e^{-1}x) = H_{\theta\omega}(x) + F_0(0), \qquad \operatorname{Var} F_0(0) = 1. \tag{5}$$

Proof. Write

$$c_k = \int_0^1 \frac{k(se_1) - 1}{s}\, ds, \qquad g(z) = c_k - \int_0^1 \frac{k(sz) - 1}{s}\, ds.$$

Radiality and the substitution $u = e^t|z|$ in (4) give $K(z) = \log(1/|z|) + g(z)$. The integral defining g is smooth at zero: $k(sz) - 1 = O(s^2)$ locally uniformly in z, and for every nonzero multi-index α,

$$\partial^\alpha g(z) = -\int_0^1 s^{|\alpha|-1}(\partial^\alpha k)(sz)\, ds.$$

This proves (i), including smoothness away from zero.

Up to the fixed Fourier-transform normalization, the spectral density of X is

$$\begin{aligned} v(\xi) &= \int_0^\infty e^{-dt}|\widehat{\phi}(e^{-t}\xi)|^2\, dt \\ &= |\xi|^{-d}\int_0^{|\xi|} s^{d-1}|\widehat{\phi}(se_1)|^2\, ds \asymp (1 + |\xi|^2)^{-d/2}. \end{aligned} \tag{6}$$

The last comparison follows from $\widehat{\phi}(0) > 0$ and rapid decay of $\widehat{\phi}$. In particular, for every $f \in C_c^\infty(\mathbb{R}^d)$,

$$\|f\|_{\mathrm{CM}}^2 = \text{constant} \times \int_{\mathbb{R}^d} \frac{|\widehat{f}(\xi)|^2}{v(\xi)}\, d\xi < \infty.$$

The Cameron–Martin theorem gives (ii). A smooth local shift is extended by a compactly supported cutoff which equals one near the bounded set under consideration.

For (iii)–(iv), let $(F_n)_{n\in\mathbb{Z}}$ be iid smooth centered stationary Gaussian fields with covariance

$$\mathrm{Cov}(F_n(x), F_n(y)) = \int_0^1 k(e^t(x-y))\,dt. \tag{7}$$

Define

$$\begin{aligned} X(x) &= \sum_{n\geq 0} F_n(e^n x), \\ Y(x) &= \sum_{n<0} \big(F_n(e^n x) - F_n(0)\big), \\ H(x) &= X(x) + Y(x). \end{aligned} \tag{8}$$

The first series converges almost surely as a Gaussian distribution and has covariance (4). More explicitly, for $\chi \in C_c^\infty$ and $s > 0$, its shell spectral densities give

$$\mathbb{E}\|\chi F_n(e^n\cdot)\|_{H^{-s}}^2 \leq C_{\chi,s}(1+n)e^{-\min(2s,d)n}.$$

The bound is summable, so the independent centered Hilbert-space increments converge almost surely in H^{-s}_{loc}. The second series converges in C^j on each compact set for every finite j: the expected C^j norm of its n-th term is $O(e^n)$ as $n \to -\infty$. Thus Y is smooth and independent of X.

To check the whole-space normalization, put

$$L(z) = \int_{-\infty}^0 \big(k(e^t z) - 1\big)\,dt = c_k - g(z).$$

Then

$$\mathrm{Cov}(Y(x), Y(y)) = L(x-y) - L(x) - L(y)$$

and the covariance kernel of H, in the distributional sense, is

$$\log\frac{1}{|x-y|} + g(x) + g(y) - c_k. \tag{9}$$

The last three terms disappear when pairing against two mean-zero test functions. Hence H modulo constants is exactly the field (1).

Conditional on Y, its restriction to any bounded set is a Cameron–Martin shift of X, by (ii). A mixture of measures each equivalent to the law of X is again equivalent to that law. The same argument includes an independent constant C. This proves (iii). Finally, reindexing (8) gives (5); the unmatched term is $F_0(0)$, and (7) gives its variance as one. □

In particular, admissibility can equivalently be defined by local absolute continuity with respect to the stationary field X. This is a convenience for the proof, not an additional axiom on $\mathfrak{m}$.

Lemma 2.2 (Smooth-average pinning)**.** *Let $\rho \in C_c^\infty(\mathbb{R}^d)$ be nonnegative with integral one, and set $H^\rho = H - (H, \rho)$. Every point has a bounded open ball B around it for which*

$$\mathrm{Law}(H^\rho|_B) \sim \mathrm{Law}(X|_B).$$

Consequently, local absolute continuity with respect to H^ρ defines exactly the admissible class in Definition 1.1.

Proof. Choose B small enough that part of $\operatorname{supp}\rho$ lies outside $\overline{B}$, with positive ρ-mass there. Choose a smooth probability density η supported at distance greater than one from both B and $\operatorname{supp}\rho$. The covariance K vanishes beyond distance one. Hence (X,η) is independent of $X|_B$, and

$$H^\eta|_B = X|_B + \big[Y - (Y,\eta) - (X,\eta)\big]|_B$$

is an independent smooth perturbation. Lemma 2.1 gives $H^\eta|_B \sim X|_B$.

Put $Z = H^\rho|_B$ and $A = (H^\rho,\eta)$, so $H^\eta|_B = Z - A$. Shifting the independent X component of H by $f \in C_c^\infty$ is a Cameron–Martin transformation; after pinning, it shifts H^ρ by $f - (f,\rho)$. Choose $f = 1$ near $\overline{B}$, adjusting it outside $\overline{B}$ so that $(f,\rho) = 0$. Thus constants are Cameron–Martin shifts of Z.

There is also a smooth Cameron–Martin shift supported near $\operatorname{supp}\eta$ which leaves Z unchanged but changes A. Therefore the conditional Gaussian variance $\operatorname{Var}(A \mid Z)$ is strictly positive: if it were zero, A would be a measurable function of Z, incompatible with equivalence under such a shift. Its conditional Gaussian density is consequently positive everywhere, and

$$\operatorname{Law}(Z,A) \sim \operatorname{Law}(Z) \otimes N(0,1).$$

Push this equivalence forward under $(z,a) \mapsto z - a$. Since all constant translates of Z have equivalent laws, it follows that $\operatorname{Law}(Z - A) \sim \operatorname{Law}(Z)$. Combining this with the remote-pinning equivalence proves the result. □

3 The recurrent Gaussian scale system

Let Ω be the two-sided sequence space of the shell fields in (8), with iid law $\mathbb{P}$. On

$$(\Omega \times \mathbb{R},\ \boldsymbol{\nu}), \qquad \boldsymbol{\nu} = \mathbb{P} \otimes dc,$$

define

$$T(\omega,c) = \big(\theta\omega,\, c + F_0(0)\big). \tag{10}$$

Here dc denotes Lebesgue measure, not a probability measure.

Lemma 3.1. *The invertible map T preserves $\boldsymbol{\nu}$ and is conservative and ergodic. Consequently, every measurable set of positive $\boldsymbol{\nu}$-measure is visited infinitely often by almost every forward orbit.*

Proof. Preservation follows by first changing variables $c \mapsto c + F_0(0)$ and then shifting the iid sequence. For the remaining claims, perform Gaussian regression within each shell:

$$F_n(x) = b(x)Z_n + R_n(x), \qquad Z_n = F_n(0), \qquad b(x) = \operatorname{Cov}(F_0(x), F_0(0)).$$

The Z_n are iid standard normals; the residual fields R_n are iid, with law ρ, and independent of the entire sequence (Z_n).

Set $S_0 = c$ and $S_{n+1} = S_n + Z_n$ in both time directions. The map from (ω,c) to the two-sided path

$$\big((S_n, R_n)\big)_{n\in\mathbb{Z}}$$

is invertible: $Z_n = S_{n+1} - S_n$ recovers every shell. Under this identification, T is the path shift of the Markov chain with invariant measure

$$\eta(ds\,dr) = ds\,\rho(dr)$$

and transition kernel

$$P((s,r),ds'\,dr') = \frac{e^{-(s'-s)^2/2}}{\sqrt{2\pi}}\,ds'\,\rho(dr'). \tag{11}$$

The measure η is sigma-finite. Every strip $[-M,M]\times\operatorname{supp}\rho$ has finite measure, these strips exhaust the state space modulo η, and the first coordinate returns to each strip almost surely by recurrence of a centered one-dimensional Gaussian random walk. Moreover, the transition law in (11) is equivalent to η from every state. In particular, an invariant set for this chain must have either zero or full η-measure.

The recurrent Markov-shift theorem therefore gives a conservative ergodic one-sided path shift; its two-sided stationary extension has the same properties. The required general implication is stated, for example, in [4, Lemma 3.1(a),(b),(d)]. Appendix A supplies a direct argument handling the two-sided extension and its marks.

In a conservative ergodic invertible system, a positive-measure set has conull orbit saturation, and almost every point of the set returns infinitely often. This gives the final assertion. When the set has infinite measure, first take a positive finite-measure subset. □

4 A logarithmic insertion forces infinite mass

Let

$$D_* = \mathbb{R}^d\setminus\{0\},\qquad A=\{x : e^{-1}<|x|<1\},\qquad A_n = e^{-n}A.$$

Both D_* and A are domains when $d\ge 2$. On D_*, set

$$\Phi_{\omega,c}(x) = H_\omega(x) + c + Q\log\frac{1}{|x|}. \tag{12}$$

For each fixed c, this is admissible on D_*: on each relatively compact set avoiding zero, its singular term is a smooth local Cameron–Martin shift.

The exact identity (5) gives

$$\Phi_{\omega,c}(e^{-1}x) + Q\log(e^{-1}) = \Phi_{T(\omega,c)}(x). \tag{13}$$

The logarithmic insertion cancels the deterministic coordinate drift exactly. Define the measurable extended nonnegative function

$$M(\omega,c) = \mathfrak{m}^{D_*}_{\Phi_{\omega,c}}(A).$$

Applying covariance to the countably many dilations $x\mapsto e^{-n}x$, then Fubini in c, yields

$$\mathfrak{m}^{D_*}_{\Phi_{\omega,c}}(A_n) = M(T^n(\omega,c)),\qquad n\ge 0, \tag{14}$$

simultaneously outside one $\boldsymbol{\nu}$-null set. Every application of the axiom here is on the punctured domain.

Proposition 4.1. *If $\mathbb{P}[\mathfrak{m}^{\mathbb{R}^d}_X\neq 0]>0$, then for $\boldsymbol{\nu}$-almost every (ω,c),*

$$\mathfrak{m}^{D_*}_{\Phi_{\omega,c}}(B(0,r)\setminus\{0\}) = \infty \qquad \text{for every } r>0. \tag{15}$$

Consequently, for every fixed $z\in\mathbb{R}^d$, the field

$$G_z(y) = X(y) + QK(y-z) \tag{16}$$

has a punctured-domain measure satisfying

$$\mathfrak{m}^{\mathbb{R}^d\setminus\{z\}}_{G_z|_{\mathbb{R}^d\setminus\{z\}}}(B(z,r)\setminus\{z\})=\infty \qquad \text{for every } r>0, \tag{17}$$

almost surely.

Proof. Fix an open ball V whose closure is contained in A. Its countably many rational translates cover $\mathbb{R}^d$. Since the global measure is nonzero with positive probability, some such translate has positive mass with positive probability. Stationarity of X and translation covariance imply

$$\mathbb{P}[\mathfrak{m}^{\mathbb{R}^d}_X(V)>0]>0.$$

By locality, this is the same event for the assignment on V. On V, the laws of $\Phi_{\omega,C}$ and X are equivalent by Lemma 2.1, where C is an independent standard normal. Applying locality on D_* gives

$$(\mathbb{P}\otimes \mathrm{Law}(C))[M>0]>0.$$

Since $\mathrm{Law}(C)$ has a positive Lebesgue density, there exist $a>0$ and $L<\infty$ such that

$$0<\boldsymbol{\nu}\{M>a,\ |c|\le L\}<\infty.$$

Lemma 3.1 and (14) show that almost every cone realization gives mass exceeding a to infinitely many of the pairwise disjoint annuli A_n. All but finitely many of these annuli lie inside any prescribed $B(0,r)$. Countable additivity proves (15).

The conclusion also holds with $c=C$, since the normal law is absolutely continuous with respect to Lebesgue measure. By locality, it is a probability-one property of the assignment to the restriction of $\Phi_{\omega,C}$ to any fixed punctured ball.

By Lemma 2.1(i),

$$X(y)+QK(y)=X(y)+Qg(y)+Q\log\frac{1}{|y|}.$$

On a bounded ball, the laws of the residual Gaussian fields $X+Qg$ and $H+C$ are equivalent. Adding their common deterministic logarithmic singularity and restricting to the punctured ball preserves equivalence. Thus the probability-one conclusion transfers to G_0 on every punctured ball. Locality on D_* gives the statement for its punctured-space assignment, simultaneously over a countable sequence of radii.

For fixed z, stationarity gives

$$G_z(z+\cdot)\ \overset{d}{=}\ X(\cdot)+QK(\cdot).$$

Translation covariance applied on $\mathbb{R}^d\setminus\{z\}$ transfers the preceding conclusion to G_z. This proves (17). No axiom has been applied to G_z on a domain containing its singularity. □

5 Rooting by subcritical Gaussian multiplicative chaos

The sole use of $Q<\sqrt{2d}$ is in this section. For the stationary field X, let

$$X_N(x)=\sum_{n=0}^{N-1}F_n(e^nx), \qquad \mu_{Q,N}(dx)=\exp\left(QX_N(x)-\frac{Q^2}{2}N\right)dx.$$

Subcritical Gaussian multiplicative chaos (GMC) gives a limit μ_Q, with L^1 convergence of the masses of every fixed bounded Borel set. This is the usual measurable GMC of X, equivalently obtained from mollifier approximations, and it satisfies

$$\mathbb{E}\mu_Q(dx) = dx. \tag{18}$$

The needed existence result is the usual logarithmic-kernel theorem in the range $Q^2 < 2d$; see [3, Section 3.4].

This chaos has full support. Indeed, for a fixed bounded nonempty open set V, the event $\{\mu_Q(V) = 0\}$ is unchanged by removing any finite number of the smooth independent shells, since their contribution is multiplication by a strictly positive smooth function. It is therefore a tail event for the independent shells. The tail zero-one law and $\mathbb{E}\mu_Q(V) = |V| > 0$ imply $\mu_Q(V) > 0$ almost surely. A countable basis yields full support simultaneously.

We use the following exact Palm identity, valid for every nonnegative measurable functional F of the distribution and the point:

$$\mathbb{E}\int_V F(X, z)\,\mu_Q(dz) = \int_V \mathbb{E}\big[F(X + QK(\cdot - z), z)\big]\,dz. \tag{19}$$

This is the Gaussian randomized-shift identity for GMC; see [3, Theorem 4, equation (7)]. The shift of X is QK, not Q^2K.

5.1 The locality issue at a Palm root

For fixed z, the shifted field $G_z = X + QK(\cdot - z)$ is singular at z. Its global assignment $\mathfrak{m}_{G_z}^{\mathbb{R}^d}$ is defined because the map is defined on all distributions, but the hypotheses do not directly grant it locality at z. We now obtain exactly the required locality from (19).

Let $\mathcal{B}$ be the countable family of open balls with rational centers and positive rational radii. Define $\mathcal{E} \subseteq \mathcal{D}'(\mathbb{R}^d)$ to be the event that, for every finite tuple $B, B_1, \dots, B_j \in \mathcal{B}$, including $j = 0$,

$$\mathfrak{m}_h^{\mathbb{R}^d}\left(B \setminus \bigcup_{i=1}^{j} B_i\right) = \mathfrak{m}_{h|_B}^{B}\left(B \setminus \bigcup_{i=1}^{j} B_i\right). \tag{20}$$

This is a countable collection of measurable scalar equalities. Locality for the admissible field X implies $\mathbb{P}[X \in \mathcal{E}] = 1$.

Apply (19) with $F(h, z) = \mathbf{1}_{\{h \notin \mathcal{E}\}}$, for a bounded set V. The left side is zero, so

$$\mathbb{P}[G_z \in \mathcal{E}] = 1 \qquad \text{for Lebesgue-almost every } z \in V. \tag{21}$$

For such a fixed z, write

$$\mu = \mathfrak{m}_{G_z}^{\mathbb{R}^d}, \qquad \widetilde{\mu} = \mathfrak{m}_{G_z|_{\mathbb{R}^d\setminus\{z\}}}^{\mathbb{R}^d\setminus\{z\}}.$$

The punctured field is admissible, so locality for $\widetilde{\mu}$ holds simultaneously on all rational balls whose closures avoid z. Together with (20), this compares μ and $\widetilde{\mu}$ on every finite Boolean piece of such balls contained in one of them.

Here is why this comparison suffices even for non-sigma-finite measures. For any open $O \subseteq \mathbb{R}^d \setminus \{z\}$, enumerate rational balls (B_i) with closures contained in O which cover O, and disjointize them:

$$E_1 = B_1, \qquad E_i = B_i \setminus \bigcup_{j<i} B_j.$$

Every E_i is among the countably many pieces in (20). Thus $\mu(E_i) = \widetilde{\mu}(E_i)$, and countable additivity gives

$$\mu(O) = \sum_i \mu(E_i) = \sum_i \widetilde{\mu}(E_i) = \widetilde{\mu}(O). \tag{22}$$

There is no appeal to uniqueness of measures from their values on a basis, which would need extra care without sigma-finiteness. There is also no assumption of joint measurability in a variable punctured domain: all Palm-transferred events above use fixed domains.

5.2 Completion of the proof

Suppose first that $\mathbb{P}[\mathfrak{m}_X^{\mathbb{R}^d} \neq 0] > 0$, so Proposition 4.1 applies. Fix a bounded nonempty open V. For almost every $z \in V$, (21) and (22), followed by (17) on a punctured ball contained in V, give

$$\mathfrak{m}^{\mathbb{R}^d}_{X+QK(\cdot-z)}(V) = \infty \quad \text{almost surely.}$$

Now apply the Palm identity to the fixed-domain measurable functional

$$F(h,z) = \mathbf{1}_{\{\mathfrak{m}_h^{\mathbb{R}^d}(V)<\infty\}}.$$

It follows that

$$\begin{aligned}\mathbb{E}\left[\mu_Q(V)\mathbf{1}_{\{\mathfrak{m}_X^{\mathbb{R}^d}(V)<\infty\}}\right] &= \int_V \mathbb{P}\left[\mathfrak{m}^{\mathbb{R}^d}_{X+QK(\cdot-z)}(V) < \infty\right] dz \\ &= 0.\end{aligned} \tag{23}$$

Since $\mu_Q(V) > 0$ almost surely, this implies

$$\mathbb{P}[\mathfrak{m}_X^{\mathbb{R}^d}(V) = \infty] = 1.$$

Taking a countable basis shows that $\mathfrak{m}_X^{\mathbb{R}^d}$ almost surely gives infinite mass to every nonempty open set.

If instead $\mathfrak{m}_X^{\mathbb{R}^d} = 0$ almost surely, locality immediately gives the same zero statement for the assignment to X on every deterministic subdomain.

Finally, let h be any admissible field on any domain U. Take a countable cover of U by relatively compact neighborhoods on which its law is absolutely continuous to that of X, using Definition 1.1, Lemma 2.1, and a countable refinement. In the zero case, locality and absolute continuity force zero mass on each member of the cover, hence on U. In the infinite case, they force infinite mass on every member of a countable basis subordinate to this cover. Every nonempty open subset of U contains one such basis element. This proves the universal alternative and completes the proof of Theorem 1.2. □

6 What the theorem does and does not assume

The auxiliary chaos is not a candidate volume measure. The parameter of μ_Q is chosen solely to insert the logarithmic singularity of coefficient Q. It is a tool for sampling a typical Q-thick point. We never assert that μ_Q has background charge Q. For ordinary local GMC with exponential parameter γ, the corresponding volume charge is

$$Q_\gamma = \frac{d}{\gamma} + \frac{\gamma}{2}. \tag{24}$$

Confusing these two roles of a parameter would invalidate the argument; they are kept separate throughout.

The threshold is sharp. For $Q > \sqrt{2d}$, the choice

$$\gamma = Q - \sqrt{Q^2 - 2d} \in (0, \sqrt{2d})$$

satisfies (24). The usual geometrically normalized local GMC is then a nonzero locally finite example with covariance (3). This follows directly by changing variables in $\varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(x)}\, dx$: the factor is $r^{\gamma Q - d - \gamma^2/2} = 1$. At $Q = \sqrt{2d}$, critical GMC supplies the corresponding example, with the critical normalization; see [5]. The ordinary mean-normalized Q-GMC used in (19) ceases to be nontrivial at this endpoint, precisely where our proof must stop.

No planar geometric input is used. The full argument uses translations, the single dilation $x \mapsto e^{-1}x$ and its iterates, Gaussian Cameron–Martin equivalence, one-dimensional random-walk recurrence, and subcritical GMC. It does not use a higher-dimensional analogue of CLE, a domain Markov decomposition for an elliptic operator, existence of an LQG metric, or uniqueness of that metric. Odd and even dimensions are treated identically.

The conclusion concerns open sets. The infinite alternative means infinite mass on every nonempty Euclidean open set. It does not assert that the measure is impossible as a sigma-finite Borel measure, nor does it determine its mass on arbitrary thin sets. The immediate exclusion is of nonzero *locally finite* measures.

A A direct check of the two-sided recurrence lemma

For completeness, we verify the dynamical assertion used in Lemma 3.1 without an unproved assumption about independent marks. Work on the two-sided path space of the stationary sigma-finite Markov chain $Y_n = (S_n, R_n)$ with kernel (11), and let T be its shift.

The finite-measure path sets $C_M = \{|S_0| \le M\}$ exhaust the space. They recur because a centered one-dimensional Gaussian random walk returns to bounded intervals. A measure-preserving transformation exhausted by finite recurrent sets is conservative; one proves this by inducing on the finite sets and applying finite-measure Poincare recurrence. The same holds for T^{-1}.

Let A be a T-invariant measurable event. Write

$$\mathcal{G}_n = \sigma(Y_k : k \ge -n), \qquad f = \mathbb{E}[\mathbf{1}_A \mid \mathcal{G}_0].$$

These conditional expectations are legitimate in the sigma-finite setting because each C_M belongs to every $\mathcal{G}_n$. Invariance gives

$$\mathbb{E}[\mathbf{1}_A \mid \mathcal{G}_n] = f \circ T^{-n}.$$

Martingale convergence, after localizing to C_M, shows that $f(T^{-n}\omega) \to \mathbf{1}_A(\omega)$ almost everywhere. If $\{\epsilon < f < 1 - \epsilon\}$ had positive measure for any $\epsilon > 0$, backward conservativity would give infinitely many visits to that set, contradicting this convergence. Hence f is indicator-valued. A conditional expectation of an indicator which is indicator-valued equals the original indicator almost everywhere, so A is measurable with respect to the future path.

Set $p(y) = \mathbb{P}_y(A)$ for the future Markov chain. Invariance and the Markov property give

$$\mathbb{E}[\mathbf{1}_A \mid Y_0, \dots, Y_n] = p(Y_n) \longrightarrow \mathbf{1}_A \quad \text{almost everywhere.}$$

Forward conservativity applied to $\{\epsilon < p(Y_0) < 1-\epsilon\}$ now implies that p is indicator-valued η-almost everywhere. Moreover, $p = Pp$. Each transition law $P(y, \cdot)$ is equivalent to η, so an indicator-valued harmonic function must be constant: if its zero and one sets both had positive η-measure, then $Pp(y)$ would lie strictly between zero and one for every y. Thus A has either zero or full path measure. This proves ergodicity of the two-sided marked shift.

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